Step-by-Step Solutions for Your HP-28S or HP-28C Calculator

Vectors and Matrices contains a variety of examples and solutions to show how you can easily solve your technical problems.

- **General Matrix Operations**
  Sum of Matrices • Matrix Multiplication • Determinant of a Matrix • Inverse of a Matrix • Transpose of a Matrix • Conjugate of a Complex Matrix • Minor of a Matrix • Rank of a Matrix • Hermitian Matrices
- **Systems of Linear Equations**
  Non-Homogeneous System • Homogeneous System • Iterative Refinement
- **Vector Spaces**
  Basis • Orthogonality • Vector Length • Normalization • Gram-Schmidt Orthogonalization • Orthonormal Basis
- **Eigenvalues**
  The Characterisitc Polynomial • Eigenvalues from Expansion • Eigenvectors • Eigenvalues from $|\lambda I - A|$
- **Least Squares**
  Straight Line Fitting • Quadratic Polynomial
- **Markov Chains**
  Steady State of a System
- **An Example: Forest Management Model and Yield**

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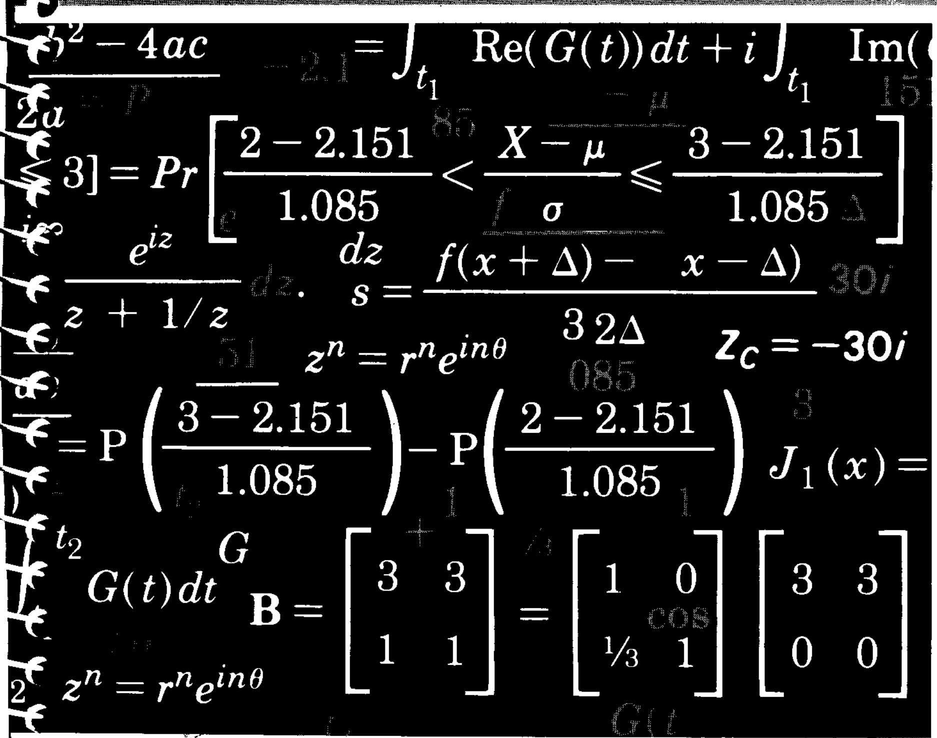
Step-by-Step Solutions
For Your HP Calculator

Vectors and Matrices



HP-28S
HP-28C

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Vectors and Matrices

Step-by-Step Solutions for Your HP-28S or HP-28C Calculator

HEWLETT PACKARD

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Welcome...

... to the HP-28S and HP-28C Step-by-Step Solution Books. These books are designed to help you get the most from your HP-28S or HP-28C calculator.

This book, *Vectors and Matrices*, provides examples and techniques for solving problems on your calculator. A variety of matrix manipulations are included to familiarize you with the many functions built into your calculator.

Before you try the examples in this book, you should be familiar with certain concepts from the owner's documentation:

- The basics of your calculator: how to move from menu to menu, how to exit graphics and edit modes, and how to use the menu to assign values to, and solve for, user variables.
- Entering numbers, programs, and algebraic expressions into the calculator.

Please review the section "How To Use This Book." It contains important information on the examples in this book.

For more information about the topics in the *Vectors and Matrices* book, refer to a basic textbook on the subject. Many references are available in university libraries and in technical and college bookstores. The examples in the book demonstrate approaches to solving certain problems, but they do not cover the many ways to approach solutions to mathematical problems.

Our thanks to Brenda C. Bowman of Oregon State University for developing the problems in this book.

Contents

How To Use This Book

Please take a moment to familiarize yourself with the formats used in this book.

Keys and Menu Selection: A box represents a key on the calculator keyboard.

[ENTER]
[1/x]
[STO]

[ARRAY]
[PLOT]
[ALGEBRA]

In many cases, a box represents a shifted key on the calculator. In the example problems, the shift key is NOT explicitly shown. (For example, [ARRAY] requires you to press the shift key, followed by the ARRAY key, found above the "A" on the left keyboard.)

The "inverse" highlight represents a menu label:

Key:	Description:
DRAW	Found in the [PLOT] menu.
ISOL	Found in the [SOLV] menu.
ABCD	A user-created name. If you created a variable by this name, it could be found in either the [USER] menu or the SOLVR menu. If you created a program by this name, it would be found in the [USER] menu.

Menus typically include more menu labels than can be displayed above the six redefinable menu keys. Press [NEXT] and [PREV] to roll through the menu options. For simplicity, [NEXT] and [PREV] are NOT shown in the examples.

Solving for a user variable within ≡SOLVR≡ is initiated by the shift key, followed by the appropriate user-defined menu key:

[ ] ≡ABCD≡.

The keys above indicate the shift key, followed by the user-defined key labeled "ABCD". Pressing these keys initiates the Solver function to seek a solution for "ABCD" in a specified equation.

The symbol [<>] indicates the cursor-menu key.

Interactive Plots and the Graphics Cursor: Coordinate values you obtain from plots using the [INS] and [DEL] digitizing keys may differ from those shown, due to small differences in the positions of the graphics cursor. The values you obtain should be satisfactory for the Solver root-finding that follows.

Display Formats and Numeric Input: Negative numbers, displayed as

```
-5
-12345.678
[[-1,-2,-3[-4,-5,-6[ ...
```

are created using the [CHS] key.

```
5 [CHS]
12345.678 [CHS]
[[1 [CHS],2 [CHS], ...
```

The examples in this book typically specify a display format for the number of decimal places. If your display is set such that numeric displays do not match exactly, you can modify your display format with the [MODE] menu and the ≡FIX≡ key within that menu. (For example, [MODE] 2 ≡FIX≡ will set the display to the FIX 2 format.)

Programming Reminders: Before you key in the programming examples in this book, familiarize yourself with the locations of programming commands that appear as menu labels. By using the menu labels to enter commands, you can speed keying in programs and avoid errors that might arise from extra spaces appearing in the programs. Remember, the calculator recognizes commands that are set off by spaces. Therefore, the arrow (→) in the command R→C (the real to complex conversion function) is interpreted differently than the arrow in the command → C (create the local variable "C").

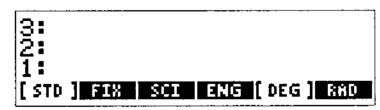
The HP-28S automatically inserts spaces around each operator as you key it in. Therefore, using the [R], [→], and [C] keys to enter the R→C command will result in the expression R → C , and, ultimately, in an error in your program. As you key in programs on the HP-28S, take particular care to avoid spaces inside commands, especially in commands that include an →.

The HP-28C does not automatically insert spaces around operators or commands as they are keyed in.

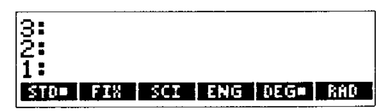
A Note About the Displays Used in This Book: The menus and screens that appear in this book show the HP-28S display. Most of the HP-28C and HP-28S screens are identical, but there are differences in the [MODE] menu and ≡SOLVR≡ screen that HP-28C users should be aware of.

For example, the first screen below illustrates the HP-28C [MODE] menu, and the second screen illustrates the same menu as it appears on the HP-28S.

HP-28C [MODE] display.



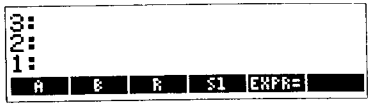
HP-28S [MODE] display.



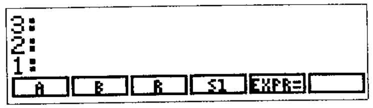
Notice that the HP-28C highlights the entire active menu item, while the HP-28S display includes a small box in the active menu item.

The screens shown below illustrate the HP-28C and HP-28S versions of the SOLVR menu.

HP-28C SOLVR display.



HP-28S SOLVR display.



Both of these screens include the Solver variables A, B, R, S1, and EXPR=. The HP-28C displays Solver variables in gray on a black background. The HP-28S prints Solver variables in black on a gray background.

User Menus: A PURGE command follows many of the examples in this book. If you do not purge all of the programs and variables after working each example, or if your USER menu contains your own user-defined variables or programs, the USER menu on your calculator may differ from the displays shown in this book. Do not be concerned if the variables and programs appear in a slightly different order on your USER menu; this will not affect the calculator's performance.

General Matrix Operations

This chapter illustrates several basic matrix manipulations found in common matrix problems, including addition, matrix multiplication, determinants, and so forth. Also included are several programs that demonstrate operations on matrix minors and rank.

Sum of Matrices

This example illustrates two methods for creating a matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & 4 & -1 & 2 \\ 1 & -3 & 2 & -2 \end{bmatrix}$$

Compute $A + B$.

CLEAR
MODE STD
<>

```
4:
3:
2:
1:
```

Key in the elements of matrix A in row order form. Put each element on the stack individually.

1 ENTER
2 ENTER
3 ENTER
4 ENTER
5 ENTER
6 ENTER
7 ENTER
8 ENTER
9 ENTER
10 ENTER
11 ENTER
12 ENTER

```
4:          9
3:         10
2:         11
1:         12
```

Key in the dimensions $\{m, n\}$ of matrix A. Remember to use a space to separate the two numbers.

{3 4} ENTER

```
4:         10
3:         11
2:         12
1:      { 3 4 }
```


Put the stack elements into the matrix.

ARRAY
→ARRY

```
1: [[ 1 2 3 4 ]
    [ 5 6 7 8 ]
    [ 9 10 11 12 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

Store the matrix in A for the next problem section.

'A STO

```
3:
2:
1:
→ARRY ARRY→ PUT GET PUTI GETI
```

Enter matrix B, using a space to separate the matrix elements. Note the two different methods used to enter the elements of A and B.

[[2 -3 0 1[0 4 -1 2
[1 -3 2 -2 ENTER

```
1: [[ 2 -3 0 1 ]
    [ 0 4 -1 2 ]
    [ 1 -3 2 -2 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

Compute the sum $A + B$.

A ENTER

```
1: [[ 1 2 3 4 ]
    [ 5 6 7 8 ]
    [ 9 10 11 12 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

+

```
1: [[ 3 -1 3 5 ]
    [ 5 10 6 10 ]
    [ 10 7 13 10 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

Matrix Multiplication

Compute the product of two matrices, The first matrix must have dimensions $k \times m$, the second matrix has dimensions $m \times n$, and the product has dimensions $k \times n$. In this example, $k = 3$, $m = 4$, and $n = 2$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ -2 & 3 \\ 5 & 4 \end{bmatrix}$$

Compute $A * D$.

Enter the 3×4 matrix A from the previous example.

A [ENTER]

```
1: [[ 1 2 3 4 ]
    [ 5 6 7 8 ]
    [ 9 10 11 12 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

Enter the 4×2 matrix D.

`[[-1 1[2 4[-2 3[5 4` [ENTER]

```
1: [[ -1 1 ]
    [ 2 4 ]
    [ -2 3 ]
→ARRY ARRY→ PUT GET PUTI GETI
```

Compute the product $A * D$.

[×]

```
1: [[ 17 34 ]
    [ 33 82 ]
    [ 49 130 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

Purge A

`'A'` [PURGE]

Determinant of a Matrix

Solve for the determinant of an $n \times n$ matrix.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & 2 \\ -1 & -2 & 3 \end{bmatrix}$$

Key in the 3×3 matrix.

[CLEAR]
`[[2 -3 1[0 5 2[-1 -2 3`
[ENTER] [ARRAY]

```
1: [[ 2 -3 1 ]
    [ 0 5 2 ]
    [ -1 -2 3 ]]
→ARRY ARRY→ PUT GET PUTI GETI
```

Compute det(A).

[DET]

```
3:
2:
1:                    49
CROSS DOT DET ABS RNRM CNRM
```

The determinant is 49.

Inverse of a Matrix

Compute the inverse of a square $n \times n$ matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Clear the stack and set the number display mode to two decimal places.

CLEAR
MODE 2 FIX

```
3:
2:
1:
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the elements of the 3×3 matrix.

`[[1 2 3[2 4 5[3 5 6`
ENTER

```
1: [[ 1.00 2.00 3.00 ]
    [ 2.00 4.00 5.00 ]
    [ 3.00 5.00 6.00 ]]
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

Compute A^{-1}.

1/x

```
1: [[ 1.00 -3.00 2.00 ]
    [ -3.00 3.00 -1.00…
    [ 2.00 -1.00 6.66E…
 STD | FIX■ | SCI | ENG | DEG | RAD■
```


Transpose of a Matrix

Compute the transpose of an $m \times n$ matrix A. A^T will be of dimension $n \times m$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Clear the display and set the mode to standard. Key in the 3×2 matrix A.

CLEAR
MODE
STD
`[[1 2[3 4[5 6` ENTER

```
1: [[ 1 2 ]
    [ 3 4 ]
    [ 5 6 ]]
 STD■ | FIX | SCI | ENG | DEG | RAD■
```

Compute A^T.

ARRAY
TRN

```
2:
1: [[ 1 3 5 ]
    [ 2 4 6 ]]
 SIZE | RDM | TRN | CON | IDN | RSD
```

A^T is a 2×3 matrix.

Conjugate of a Complex Matrix

Compute the conjugate conj(A) of the complex matrix A.

$$A = \begin{bmatrix} 1+3i & i \\ 3 & 2-4i \end{bmatrix}$$

[CLEAR]

```
3:
2:
1:
 SIZE | RDM | TRN | CON | IDN | RSD
```

Key in the elements individually in row order form. Each pair represents (real part, imaginary part). Note the commas in the keystrokes below may be used alternately with spaces.

`(1,3` [ENTER]

```
3:
2:
1:                        (1,3)
 SIZE | RDM | TRN | CON | IDN | RSD
```

`(0,1` [ENTER]

```
3:
2:                        (1,3)
1:                        (0,1)
 SIZE | RDM | TRN | CON | IDN | RSD
```

`(3,0` [ENTER]

```
3:                        (1,3)
2:                        (0,1)
1:                        (3,0)
 SIZE | RDM | TRN | CON | IDN | RSD
```

`(2,-4` [ENTER]

```
3:                        (0,1)
2:                        (3,0)
1:                       (2,-4)
 SIZE | RDM | TRN | CON | IDN | RSD
```

Key in the dimensions of the matrix.

`{2 2}` [ENTER]

```
3:                        (3,0)
2:                       (2,-4)
1:                      { 2 2 }
 SIZE | RDM | TRN | CON | IDN | RSD
```


Place the stack elements in an array.

[ARRAY]
[→ARRY]

```
2:
1: [[ (1,3) (0,1) ]
    [ (3,0) (2,-4) ]]
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Compute the conjugate.

[CONJ]

```
2:
1: [[ (1,-3) (0,-1) ]
    [ (3,0) (2,4) ]]
 R→C | C→R | RE | IM | CONJ | NEG
```

Minor of a Matrix

The minor M_{ij} is formed by removing row i and column j from matrix A, then calculating the det (M_{ij}). This problem section develops a program to form the minor of any $n \times n$ matrix.

Key in the following program, and store it as ROW. ROW will be used as a subroutine used to remove a row or column from a matrix.

Program:	Comments:
« SWAP ARRY→ LIST→	Swap matrix into level one, then separate the matrix into individual elements and its dimension.
DROP	Drop the number of items in the list.
→ n m «	Save the row and column in n and m.
n DUP m × 2 +	Compute offset to row (col) number on stack.
ROLL - m × →LIST	Place $(n-i)*m$ elements into list.
→ list « m DROPN	Drop row i (col j) from stack.
list » LIST→	Separate temporary list into individual elements.
DROP	Drop number of list elements.
n 1 - m 2 →LIST	Reconstruct matrix with row (col) removed.
→ARRY »	

[ENTER] 'ROW [STO] Key in the following program and store it as the user program MINOR. MINOR utilizes the subroutine ROW to remove a row, and then a column, from the matrix.

Program:	Comments:
3 ROLLD	Roll down the matrix and row i.
ROW TRN	Remove row i and transpose for column removal.
SWAP ROW TRN	Remove column j and transpose back.

[ENTER] 'MINOR [STO]

Example: Compute M_{23} of the following matrix.

$$A = \begin{bmatrix} 2 & -3 & 4 & -4 \\ 6 & 5 & 2 & -1 \\ 1 & 0 & 3 & -2 \\ 0 & -5 & 3 & -6 \end{bmatrix}$$

Enter the matrix.

[CLEAR] [<>]
[[2 -3 4 -4[6 5 2 -1[1 0 3 -2[0 -5 3 -6 [ENTER]

```
1: [[ 2 -3 4 -4 ]
    [ 6 5 2 -1 ]
    [ 1 0 3 -2 ]
    [ 0 -5 3 -6 ]]
```

Enter the row and column to be removed.

2 [ENTER] 3 [ENTER]

```
4:
3: [[ 2 -3 4 -4 ]  [ 6…
2:                    2
1:                    3
```

Compute M_{23}.

[USER] [MINO]

```
1: [[ 2 -3 -4 ]
    [ 1 0 -2 ]
    [ 0 -5 -6 ]]
MINO ROW
```

Compute the minor det (M_{ij}).

[ARRAY] [DET]

```
3:
2:
1:                  -18
CROSS DOT DET ABS RNRM CNRM
```

The minor det(M_{23}) is -18.

Compute Rank

The dimension of the largest square submatrix whose determinant is non-zero is called the rank of the matrix. The rank is the maximum number of linearly independent row and column vectors.

Example: Find the rank of matrix A.

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 0 & 5 & -1 \\ 12 & -4 & -1 \end{bmatrix}$$

Program MDET is used to obtain the determinant of an arbitrary matrix minor. This program uses the program MINOR from page 21.

Program:	**Comments:**
« 3 PICK	Duplicate the matrix.
3 ROLLD MINOR	Produce the matrix minor.
DET »	Compute the minor determinant.

[ENTER] 'MDET [STO]

Key in the matrix.

[<>] [CLEAR]
[[4 2 -1[0 5 -1
[12 -4 -1 [ENTER]

```
2:
1: [[ 4 2 -1 ]
    [ 0 5 -1 ]
    [ 12 -4 -1 ]]
```

Make a copy of the matrix and compute the determinant to determine whether the rank = n = 3.

[ENTER] [ARRAY] [DET]

```
3:
2: [[ 4 2 -1 ]  [ 0 5 …
1:          .000000000048
CROSS DOT DET ABS RNRM CNRM
```

Det(A) is approximately 0, so rank(A) is not equal to 3.

Discard det(A).

[DROP] [<>]

```
2:
1: [[ 4 2 -1 ]
    [ 0 5 -1 ]
    [ 12 -4 -1 ]]
```


Compute the minor for the 2 × 2 submatrices of A until a minor is found that is not equal to 0.

Compute det M_{11}.

1 [ENTER] [ENTER]
[USER] [MDET]

```
3:
2: [[ 4 2 -1 ]  [ 0 5 …
1:                    -9
MDET MINO ROW
```

Det(M_{11}) is equal to -9, so rank(A) is equal to 2.

If you wish, purge programs ROW, MDET and MINO before continuing.

{'ROW' 'MDET' 'MINOR' [PURGE]

Hermitian Matrices

Determine whether a matrix is Hermitian. A square matrix with real or complex elements is Hermitian if the matrix is equal to its conjugate transpose.

Example: Determine whether the 4×4 matrix A is Hermitian.

$$A = \begin{bmatrix} 1 & 2-i & 2 & -3+i \\ 2+i & 3 & i & 3 \\ 2 & -i & 4 & 1-i \\ 3-i & 3 & 1+i & 0 \end{bmatrix}$$

Put the elements of A on the stack individually.

CLEAR <>
1 ENTER
(2,-1 ENTER
2 ENTER
(-3,1 ENTER

```
4:              1
3:         (2,-1)
2:              2
1:         (-3,1)
```

(2,1 ENTER
3 ENTER
(0,1 ENTER
3 ENTER

```
4:          (2,1)
3:              3
2:          (0,1)
1:              3
```

2 ENTER
(0,-1 ENTER
4 ENTER
(1,-1 ENTER

```
4:              2
3:         (0,-1)
2:              4
1:         (1,-1)
```

(3,-1 ENTER
3 ENTER
(1,1 ENTER
0 ENTER

```
4:         (3,-1)
3:              3
2:          (1,1)
1:              0
```

Enter the dimensions of A.

{4 4 ENTER

```
4:              3
3:          (1,1)
2:              0
1:          { 4 4 }
```

Place the elements into the matrix.

ARRY
→ARRY

```
1: [[ (1,0) (2,-1) (2,…
    [ (2,1) (3,0) (0,1…
    [ (2,0) (0,-1) (4,…
→ARRY ARRY→ PUT GET PUTI GETI
```

You can view the entire matrix to check for correctness using EDIT or VIEW↓

Make a copy of the matrix.

ENTER

```
1: [[ (1,0) (2,-1) (2,…
    [ (2,1) (3,0) (0,1…
    [ (2,0) (0,-1) (4,…
→ARRY ARRY→ PUT GET PUTI GETI
```

Compute the conjugate transpose. Since A is complex, function TRN performs both the transpose and the conjugation.

TRN

```
1: [[ (1,0) (2,-1) (2,…
    [ (2,1) (3,0) (0,1…
    [ (2,0) (0,-1) (4,…
SIZE RDM TRN CON IDN RSD
```

Test conj(A^T) and A for equivalency. If A is Hermitian, conj(A^T) and A will be equal, and SAME will return a true flag(1).

TEST SAME

```
3:
2:
1:              0
AND OR XOR NOT SAME ==
```

Matrix A is not Hermitian.

Systems of Linear Equations

One of the most frequently used and fundamental applications of matrices arises from the need to solve a system of m linear equations in n unknowns. The HP-28S and HP-28C can be used to find solutions to both non-homogeneous and homogeneous systems of the form $AX = B$.

Non-Homogeneous System

Solve a system of linear equations of the form $AX = B$.

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 + 3x_5 &= 1 \\ 3x_1 + 2x_2 - 4x_3 - 3x_4 - 8x_5 &= 2 \\ 2x_1 - x_2 + 2x_3 + 2x_4 + 5x_5 &= 3 \end{aligned}$$

Clear the stack and set the display mode to two decimal places.

[CLEAR]
[MODE] 2 [FIX]

```
3:
2:
1:
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the coefficients of the system of equations.

[[1 1 -2 1 3[3 2 -4 -3
-8[2 -1 2 2 5 [ENTER]

```
1: [[ 1.00 1.00 -2.00 …
    [ 3.00 2.00 -4.00 …
    [ 2.00 -1.00 2.00 …
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

Store matrix A.

'A [STO]

```
3:
2:
1:
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the elements of B.

[[1[2[3 [ENTER]

```
1: [[ 1.00 ]
    [ 2.00 ]
    [ 3.00 ]]
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

Store matrix B.

'B [STO]

```
3:
2:
1:
 STD | FIX■ | SCI | ENG | DEG | RAD■
```

To solve for X, use the method

$$X = \frac{A^T B}{A^T A}$$

Compute A^T.

[ARRAY]
A [ENTER]

```
1: [[ 1.00 1.00 -2.00 …
   [ 3.00 2.00 -4.00 …
   [ 2.00 -1.00 2.00 …
→ARRY ARRY→ PUT GET PUTI GETI
```

[TRN]

```
1: [[ 1.00 3.00 2.00 ]
   [ 1.00 2.00 -1.00 ]
   [ -2.00 -4.00 2.00…
SIZE RDM TRN CON IDN RSD
```

Multiply by B.

B [×]

```
1: [[ 13.00 ]
   [ 2.00 ]
   [ -4.00 ]
SIZE RDM TRN CON IDN RSD
```

Compute A^T.

A [ENTER] [TRN]

```
1: [[ 1.00 3.00 2.00 ]
   [ 1.00 2.00 -1.00 ]
   [ -2.00 -4.00 2.00…
SIZE RDM TRN CON IDN RSD
```

Multiply by A.

A [×]

```
1: [[ 14.00 5.00 -10.0…
   [ 5.00 6.00 -12.00…
   [ -10.00 -12.00 24…
SIZE RDM TRN CON IDN RSD
```

Divide A^TB by A^TA.

[÷]

```
1: [[ 1.12 ]
   [ 1.24 ]
   [ 0.80 ]
SIZE RDM TRN CON IDN RSD
```

[VIEW↑] and [VIEW↓] can be used to display all of the elements. They are $x_1 = 1.12$, $x_2 = 1.24$, $x_3 = 0.80$, $x_4 = -0.08$, and $x_5 = 0.11$.

Purge matrices A and B.

{'A''B' [PURGE]

Homogeneous System

Solve a homogeneous system of linear equations of the form $AX = 0$.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + 6x_2 + x_3 &= 0 \\ 3x_1 - 4x_2 + 8x_3 &= 0 \end{aligned}$$

The following program takes a stack of vectors representing homogeneous simultaneous equations and transforms the vectors in the stack to upper triangular form. After keying the program in, store it in UT.

Program:	**Comments:**
« DUP SIZE LIST→	Save number of elements
DROP → s	as s.
« s 2	For j = s (down) to 2, transform the bottom j − 1 vectors.
FOR j s j −	m = s − j + 1
1 + → m	
« 1 j 1 −	Loop for i = 1 to j − 1
FOR i i ROLL j PICK m 1	
→LIST DUP2 GET 4 PICK	Transform the vectors.
ROT GET SWAP ÷ × −	
i ROLLD NEXT » −1 STEP »	

[ENTER] 'UT [STO]

Set the display mode to one decimal place.

[CLEAR]
[MODE] 1 [FIX]

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Key in the coefficients.

[[1 -2 3[2 6 1[3 -4 8
[ENTER]

```
1: [[ 1.0 -2.0 3.0 ]
   [ 2.0 6.0 1.0 ]
   [ 3.0 -4.0 8.0 ]]
STO FIX■ SCI ENG DEG RAD■
```

Store the matrix in *ARR* for a verification at the end of the problem.

'ARR [STO]

```
3:
2:
1:
 STD | FIX | SCI | ENG | DEG | RAD
```

Edit matrix *ARR* to reduce to row echelon form.

[USER]
[ARR]

```
1: [[ 1.0 -2.0 3.0 ]
    [ 2.0 6.0 1.0 ]
    [ 3.0 -4.0 8.0 ]]
 ARR | UT |  |  |  |
```

Use [EDIT] mode and the [DEL] key to remove the outer brackets of the array *ARR* and place the rows into three independent row vectors. After removing the left- and right-most braces, the edited rows are [ENTER]ed:

```
[ 1 -2 3 ]
[ 2  6 1 ]
[ 3 -4 8 ]  [ENTER]
```

```
3:        [ 1.0 -2.0 3.0 ]
2:         [ 2.0 6.0 1.0 ]
1:        [ 3.0 -4.0 8.0 ]
 ARR | UT |  |  |  |
```

Now transform the matrix to upper triangular form.

[UT]

```
3:        [ 1.0 -2.0 3.0 ]
2:      [ 0.0 10.0 -5.0 ]
1:         [ 0.0 0.0 0.0 ]
 ARR | UT |  |  |  |
```


The matrix is now in row echelon form, so the system of three transformed equations is ready to be solved. The matrix represents the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 10x_2 - 5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Drop the equation 0=0.

[DROP]

```
3:
2:        [ 1.0 -2.0 3.0 ]
1:      [ 0.0 10.0 -5.0 ]
 ARR | UT |  |  |  |
```

Enter the equation from row 2.

'10×X2 - 5×X3=0 [ENTER]

```
3:        [ 1.0 -2.0 3.0 ]
2:      [ 0.0 10.0 -5.0 ]
1:          '10*X2-5*X3=0'
 ARR | UT |  |  |  |
```

Solve the equation in terms of x_3.

'X3 [ENTER]

```
3:      [ 0.0 10.0 -5.0 ]
2:          '10*X2-5*X3=0'
1:                    'X3'
 ARR | UT |  |  |  |
```

Isolate the term x_3.

[ALGEBRA]
[ISOL]

```
3:        [ 1.0 -2.0 3.0 ]
2:      [ 0.0 10.0 -5.0 ]
1:                '10*X2/5'
 TAYLR | ISOL | QUAD | SHOW | OBGET | EXGET
```

Collect terms.

[COLCT]

```
3:        [ 1.0 -2.0 3.0 ]
2:      [ 0.0 10.0 -5.0 ]
1:                  '2*X2'
 COLCT | EXPAN | SIZE | FORM | OBSUB | EXSUB
```

The solution is $x_3 = 2*x_2$. Remove row 2 to solve row 1.

[DROP]
[DROP]

```
3:
2:
1:        [ 1.0 -2.0 3.0 ]
 COLCT | EXPAN | SIZE | FORM | OBSUB | EXSUB
```

Enter the equation for row 1, making the substitution for x_3.

'X1 − 2 × X2 + 6 × X2
[ENTER]

```
3:
2:      [ 1.0 -2.0 3.0 ]
1:      'X1-2*X2+6*X2'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Solve for x_1.

'X1 [ENTER]

```
3:      [ 1.0 -2.0 3.0 ]
2:      'X1-2*X2+6*X2'
1:                'X1'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Isolate the term.

[ISOL]

```
3:
2:      [ 1.0 -2.0 3.0 ]
1:      '-(6*X2)+2*X2'
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Collect terms.

[COLCT]

```
3:
2:      [ 1.0 -2.0 3.0 ]
1:           '-(4*X2)'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

The result is $x_1 = -4 * x_2$. A solution is $x_1 = -4, x_2 = 1, x_3 = 2$. Verify this 3×1 solution vector X. Key in vector X.

[[−4[1[2 [ENTER]

```
1: [[ -4.0 ]
    [ 1.0 ]
    [ 2.0 ]]
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Put the coefficient matrix ARR on the stack.

[USER]
[ARR]

```
1: [[ 1.0 -2.0 3.0 ]
    [ 2.0 6.0 1.0 ]
    [ 3.0 -4.0 8.0 ]]
ARR UT
```

Swap the positions of ARR and X.

[SWAP]

```
1: [[ -4.0 ]
    [ 1.0 ]
    [ 2.0 ]]
ARR UT
```


Multiply $ARR * X$.

[×]

```
1: [[ 0.0 ]
    [ 0.0 ]
    [ 0.0 ]]
ARR UT
```

$ARR * X = 0$. Thus X is a verified solution to the system.

Program UT will be used in a later problem section. Purge matrix ARR.
[PURGE]

Iterative Refinement

Due to rounding errors, in some cases the numerically calculated solution Z is not precisely the solution to the original system $AX = B$. In many applications, Z may be an adequate solution. When additional accuracy is desired, the computed solution Z can be improved by the method of iterative refinement. This method uses the residual error associated with a solution to modify the solution.

Solve the system of linear equations $AX = B$.

$$A = \begin{bmatrix} 33 & 16 & 72 \\ -24 & -10 & -57 \\ -8 & -4 & -17 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Clear the display and the set the standard display mode.

CLEAR
MODE STD

```
3:
2:
1:
STD FIX SCI ENG DEG RAD
```

Solve for $AX = B$ and improve the accuracy by iterative refinement using residual corrections. Key in the coefficient matrix.

[[33 16 72[-24 -10 -57
[-8 -4 -17 ENTER

```
1: [[ 33 16 72 ]
    [ -24 -10 -57 ]
    [ -8 -4 -17 ]]
STD FIX SCI ENG DEG RAD
```

Store matrix A.

'A STO

```
3:
2:
1:
STD FIX SCI ENG DEG RAD
```

Key in the constant matrix.

[[0[0[1 ENTER

```
1: [[ 0 ]
    [ 0 ]
    [ 1 ]]
STD FIX SCI ENG DEG RAD
```


Store matrix B.

'B STO

```
3:
2:
1:
STD FIX SCI ENG DEG RAD
```

Compute $Z = B/A$.

USER
B

```
1: [[ 0 ]
    [ 0 ]
    [ 1 ]]
B A UT
```

A

```
1: [[ 33 16 72 ]
    [ -24 -10 -57 ]
    [ -8 -4 -17 ]]
B A UT
```

÷

```
1: [[ -31.9999999989 ]
    [ 25.4999999991 ]
    [ 8.99999999969 ]]
B A UT
```

Store the approximate 3×1 solution matrix Z.

'Z STO

```
3:
2:
1:
Z B A UT
```

Compute the Residual Error Matrix R, where $R = B - AZ$. The function RSD calculates R using extended precision.

B
A
Z

```
1: [[ -31.9999999989 ]
    [ 25.4999999991 ]
    [ 8.99999999969 ]]
Z B A UT
```

Solve using the RSD function.

ARRAY
RSD

```
1: [[ .00000000042 ]
    [ -.00000000027 ]
    [ -.00000000007 ]]
SIZE RDM TRN CON IDN RSD
```

Store matrix R.

'R STO

```
3:
2:
1:
SIZE RDM TRN CON IDN RSD
```

Find the actual error $E = X - Z = (B - AZ)/A = R/A$.

USER
R
A
÷

```
1: [[ -1.09999999997E-…
    [ 8.99999999977E-1…
    [ 3.09999999992E-1…
 R  Z  B  A  UT
```

Compute the corrected solution $X = Z + E$.

Z
+

```
1: [[ -32 ]
    [ 25.5 ]
    [ 9 ]]
 R  Z  B  A  UT
```

X = the corrected solution.

Purge the variables R, Z, B, and A if desired.

{ 'R' 'Z' 'B' 'A' PURGE

Vector Spaces

Vector spaces are widely used in mathematics, physics, and engineering to represent physical properties such as magnitude and direction within a geometric system. Several important vector operations can be performed easily using the built-in functions of the ARRAY menu.

Basis

A basis is a set of n linearly independent vectors that span the vector space V_n(R).

Determine whether the vectors X_1, X_2, and X_3 form a basis that spans V_3(R).

$$X_1 = [\ 1 \quad 1\ 2\ ]$$
$$X_2 = [\ 3 \quad 2\ 4\ ]$$
$$X_3 = [\ 1\ -3\ 1\ ]$$

Clear the stack and set the standard display mode.

CLEAR
MODE ≡STD≡

```
3:
2:
1:
STD■ FIX SCI ENG DEG RAD■
```

Key in the three vectors as a 3 × 3 matrix A and make two copies.

[[1 1 2[3 2 4[1 -3 1
ENTER ENTER

```
1: [[ 1 1 2 ]
    [ 3 2 4 ]
    [ 1 -3 1 ]]
STD■ FIX SCI ENG DEG RAD■
```

Store matrix A. A will be used in the next problem section.

'A STO

```
1: [[ 1 1 2 ]
    [ 3 2 4 ]
    [ 1 -3 1 ]]
STD■ FIX SCI ENG DEG RAD■
```

Compute det(A).

ARRAY
≡DET≡

```
3:
2:
1:          -7.00000000004
CROSS DOT DET ABS RNRM CNRM
```

Det(A) = −7. Thus A is non-singular, and the three row vectors are linearly independent and form a basis.

Orthogonality

Two vectors are mutually orthogonal if their inner product equals zero.

Determine which of the vectors from the previous problem are mutually orthogonal.

CLEAR

```
3:
2:
1:
CROSS DOT DET ABS RNRM CNRM
```

Recall matrix A to the stack.

A ENTER

```
1: [[ 1 1 2 ]
    [ 3 2 4 ]
    [ 1 -3 1 ]]
CROSS DOT DET ABS RNRM CNRM
```

Use EDIT to remove the outer brackets of the array A and form three row vectors. After removing the left and rightmost braces with DEL, the edited rows are ENTERed:

[1 1 2]
[3 2 4]
[1 -3 1] ENTER

```
3:          [ 1 1 2 ]
2:          [ 3 2 4 ]
1:         [ 1 -3 1 ]
CROSS DOT DET ABS RNRM CNRM
```

Note: Two utility routines for modifying a two-dimensional array to its row components and vice versa are shown at the end of this section. These routines can be used as alternatives for the editing shown above.

The third vector is X_3.

'X3 STO

```
3:
2:          [ 1 1 2 ]
1:          [ 3 2 4 ]
CROSS DOT DET ABS RNRM CNRM
```

The second vector is X_2.

'X2 STO

```
3:
2:
1:          [ 1 1 2 ]
CROSS DOT DET ABS RNRM CNRM
```

The first vector is X_1.

'X1 STO

```
3:
2:
1:
CROSS DOT DET ABS RNRM CNRM
```

Compute the inner products.

X1 [ENTER]
X2 [ENTER]

```
3:
2:                [ 1 1 2 ]
1:                [ 3 2 4 ]
CROSS DOT DET ABS RNRM CNRM
```

[DOT]

```
3:
2:
1:                       13
CROSS DOT DET ABS RNRM CNRM
```

$X_1 \cdot X_2 = 13$. These rows are not orthogonal.

[DROP]
X2 [ENTER]
X3 [ENTER]

```
3:
2:                [ 3 2 4 ]
1:               [ 1 -3 1 ]
CROSS DOT DET ABS RNRM CNRM
```

[DOT]

```
3:
2:
1:                        1
CROSS DOT DET ABS RNRM CNRM
```

$X_2 \cdot X_3 = 1$. These rows are not orthogonal.

[DROP]
X1 [ENTER]
X3 [ENTER]

```
3:
2:                [ 1 1 2 ]
1:               [ 1 -3 1 ]
CROSS DOT DET ABS RNRM CNRM
```

[DOT]

```
3:
2:
1:                        0
CROSS DOT DET ABS RNRM CNRM
```

$X_1 \cdot X_3 = 0$. These two vectors are mutually orthogonal.

Matrix Utility Programs

Several problem sections up to this point have included use of [EDIT] mode to reduce a matrix to its row elements. The following utility programs can be used as alternatives for changing a matrix to its row elements and vice versa.

Note: Your [USER] menu may look different than those displayed below. This will not affect the performance of your calculator.

Program ROW→ below takes a stack of n row vectors and the number n in level one and returns the matrix combining those n row vectors.

```
« OVER SIZE LIST→ DROP
→  n m « 0 n 1 -
FOR i i m × n i - +
ROLL ARRY→ DROP NEXT
n m 2 →LIST →ARRY » »
```

[ENTER] [< >]

```
1: « OVER SIZE LIST→
   DROP → n m « 0 n 1 -
   FOR i i m * n i - +
   ROLL ARRY→ DROP NEXT
```

After keying in the program above, store the program and put the rows of array A in matrix form.

'ROW→ [STO]

[1,1,2]
[3,2,4]
[1,-3,1] [ENTER]
3 [USER] [ROW→]

```
1: [[ 1 1 2 ]
    [ 3 2 4 ]
    [ 1 -3 1 ]]
ROW→ UT
```

Program →ROW below takes a matrix and separates it into individual rows on the stack.

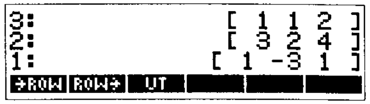
```
« ARRY→ LIST→ DROP
→  n m « 1 n FOR i m 1
→LIST →ARRY n i -
m × i + ROLLD NEXT » »
```

[ENTER] [< >]

```
1: « ARRY→ LIST→ DROP →
   n m « 1 n FOR i m 1
   →LIST →ARRY n i - m
   * i + ROLLD NEXT » »
```

After keying in the program above, store the program and convert the matrix from above back to row form.

'→ROW [STO]
[USER] →ROW



Vector Length

Find the length of vector X_1 (from the previous problem section), denoted by

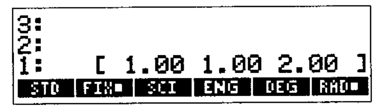
$$||X_1|| = \sqrt{X_1 \cdot X_1}$$

Clear the stack and set the display mode to two decimal places.

[CLEAR]
[MODE] 2 FIX

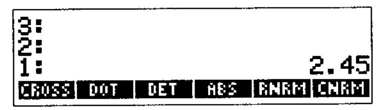
Recall X_1 from the previous problem. Since X_1 was stored, you may alternatively use [USER] X1 .

X1 [ENTER]



Function ABS returns the Frobenius norm of an array, which is equivalent to the length of a vector.

[ARRAY]
ABS



$||X_1|| = 2.45.$

Normalization

To normalize a vector X into its unique unit vector U, divide each component of X by $||X||$. We will normalize X_1. Vectors X_1, X_2, and X_3 are from the section entitled "Orthogonality."

Enter a program that computes $X/||X||$.

CLEAR
« DUP ABS INV × » ENTER

```
3:
2:
1:      « DUP ABS INV * »
CROSS DOT DET ABS RNRM CNRM
```

Store program NORM.

'NORM STO

```
3:
2:
1:
CROSS DOT DET ABS RNRM CNRM
```

Enter the vector to be normalized.

USER
X1

```
3:
2:
1:    [ 1.00 1.00 2.00 ]
NORM X1 X2 X3 →ROW ROW→
```

Normalize the vector.

NORM

```
3:
2:
1:    [ 0.41 0.41 0.82 ]
NORM X1 X2 X3 →ROW ROW→
```

The result is U_1 = [0.41 0.41 0.82].

Normalize vector X_2.

X2

```
3:
2:    [ 0.41 0.41 0.82 ]
1:    [ 3.00 2.00 4.00 ]
NORM X1 X2 X3 →ROW ROW→
```

NORM

```
3:
2:    [ 0.41 0.41 0.82 ]
1:    [ 0.56 0.37 0.74 ]
NORM X1 X2 X3 →ROW ROW→
```

The result is U_2 = [0.56 0.37 0.74].

Finally, normalize vector X_3.

X3

```
3:     [ 0.41 0.41 0.82 ]
2:     [ 0.56 0.37 0.74 ]
1:   [ 1.00 -3.00 1.00 ]
NORM X1 X2 X3 →ROW ROW→
```

NORM

```
3:     [ 0.41 0.41 0.82 ]
2:     [ 0.56 0.37 0.74 ]
1:   [ 0.30 -0.90 0.30 ]
NORM X1 X2 X3 →ROW ROW→
```

The result is U_3 = [0.30 −0.90 0.30].

You can purge the programs →ROW and ROW→ if you wish, but these programs are useful tools for matrix manipulation.

Gram-Schmidt Orthogonalization

Form an orthogonal basis that spans $V_3(R)$ using the Gram-Schmidt process. Given that X_1, X_2, and X_3 form a basis, then the vectors Y_1, Y_2, and Y_3 form an orthogonal basis by the following process.

$$Y_1 = X_1$$

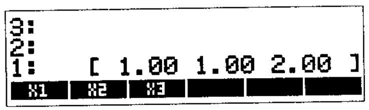
$$Y_2 = X_2 - \left(\frac{Y_1 \cdot X_2}{Y_1 \cdot Y_1} * Y_1\right)$$

$$Y_3 = X_3 - \left(\frac{Y_2 \cdot X_3}{Y_2 \cdot Y_2} * Y_2\right) - \left(\frac{Y_1 \cdot X_3}{Y_1 \cdot Y_1} * Y_1\right)$$

Vectors X_1, X_2, and X_3 are from the section entitled "Orthogonality." Remember, your [USER] menu may differ from those shown below.

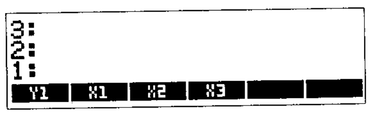
Calculate Y_1.

[CLEAR]
[USER] X1



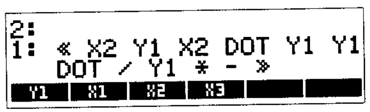
Store Y_1.

'Y1 [STO]



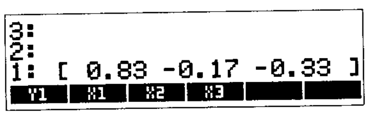
Write a program to calculate Y_2.

« X2 Y1 X2 DOT Y1 Y1
DOT ÷ Y1 × − »
[ENTER]



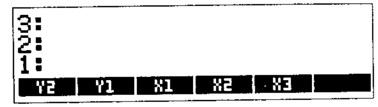
Execute the program.

[EVAL]



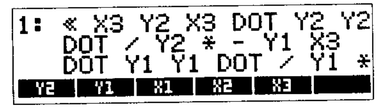
Y_2 = [0.83 −0.17 −0.33]. Store Y_2.

'Y2 [STO]



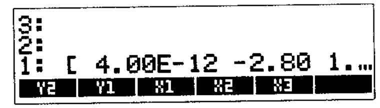
Write a program to calculate Y_3.

« X3 Y2 X3 DOT Y2 Y2
DOT ÷ Y2 × − Y1 X3
DOT Y1 Y1 DOT ÷ Y1
× − » [ENTER]



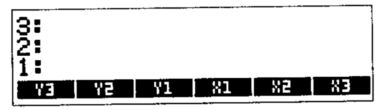
Execute the program.

[EVAL]



Y_3 = [4.00E−12 −2.80 1.40]. Store Y_3.

'Y3 [STO]



The vectors Y_1, Y_2, and Y_3 form an orthogonal basis.

Generalized Gram-Schmidt Orthogonalization Routine

The program below is a generalized routine for finding an orthogonal basis for an arbitrary list of vectors.

```
« DUP SIZE LIST→ DROP
DUP DUP 2 + ROLLD →LIST
→ M « 2 SWAP FOR n M
n GET 1 n 1 - FOR i M i
GET DUP DUP2 DOT INV ×
SWAP 3 PICK DOT × -
NEXT n M SWAP ROT PUT 'M'
STO NEXT M LIST→
DROP » » [ENTER] [<>]
```

```
1: « DUP SIZE LIST→
   DROP DUP DUP 2.00 +
   ROLLD →LIST → M «
   2.00 SWAP FOR n M n
```

Store the program as GSO and use it to form an orthogonal basis for the three vectors in the previous example.

```
'GSO [STO]
[1,1,2]
[3,2,4]
[1,-3,1] [USER] [GSO]
```

```
3:    [ 1.00 1.00 2.00 ]
2: [ 0.83 -0.17 -0.33 ]
1: [ 4.00E-12 -2.80 1.…
GSO | Y3 | Y2 | Y1 | X1 | X2
```


Orthonormal Basis

Form an orthonormal basis G_i of orthogonal unit vectors that spans $V_3(R)$. Vectors Y_1, Y_2, and Y_3 and program NORM are from the two previous problem sections.

$$G_i = \frac{Y_i}{||Y_i||}$$

Your user menu may differ from those shown here.

Calculate G_1.

```
[CLEAR]
[USER] [Y1]
```

```
3:
2:
1:    [ 1.00 1.00 2.00 ]
NORM | Y3 | Y2 | Y1 | X1 | X2
```

Execute the normalization program (NORM) from the section entitled "Normalization."

```
[NORM]
```

```
3:
2:
1:    [ 0.41 0.41 0.82 ]
NORM | Y3 | Y2 | Y1 | X1 | X2
```

Store the result in G_1.

```
'G1 [STO]
```

```
3:
2:
1:
G1 | NORM | Y3 | Y2 | Y1 | X1
```

Calculate G_2.

```
[Y2]
```

```
3:
2:
1: [ 0.83 -0.17 -0.33 ]
G1 | NORM | Y3 | Y2 | Y1 | X1
```

Compute the norm.

```
[NORM]
```

```
3:
2:
1: [ 0.91 -0.18 -0.37 ]
G1 | NORM | Y3 | Y2 | Y1 | X1
```

Store the result in G_2.

'G2 STO

```
3:
2:
1:
 G2 | G1 |NORM| Y3 | Y2 | Y1
```

Calculate G_3.

Y3

```
3:
2:
1: [ 4.00E-12 -2.80 1.…
 G2 | G1 |NORM| Y3 | Y2 | Y1
```

Compute the norm.

NORM

```
3:
2:
1: [ 1.28E-12 -0.89 0.…
 G2 | G1 |NORM| Y3 | Y2 | Y1
```

Store the result in G_3.

'G3 STO

```
3:
2:
1:
 G3 | G2 | G1 |NORM| Y3 | Y2
```

Verify that all three vectors are mutually orthogonal.

G1
G2

```
3:
2:    [ 0.41 0.41 0.82 ]
1: [ 0.91 -0.18 -0.37 ]
 G3 | G2 | G1 |NORM| Y3 | Y2
```

Compute the dot product ($G_1 \cdot G_2$).

ARRAY
DOT

```
3:
2:
1:            -8.98E-12
CROSS| DOT | DET | ABS |RNRM|CNRM
```

$G_1 \cdot G_2 \approx 0$.

Compute the dot product ($G_2 \cdot G_3$).

DROP
USER
G2
G3
ARRAY
DOT

```
3:
2:
1:             5.18E-13
CROSS| DOT | DET | ABS |RNRM|CNRM
```


$G_2 \cdot G_3 \approx 0$.

Compute the dot product ($G_1 \cdot G_3$).

DROP
USER
G1
G3
ARRAY
DOT

```
3:
2:
1:             2.97E-12
CROSS| DOT | DET | ABS |RNRM|CNRM
```

$G_1 \cdot G_3 \approx 0$.

All three dot products are approximately equal to zero and, therefore, the three vectors are mutually orthogonal.

Now verify that they form a basis. Combine the three vectors into one array by placing the elements on the stack and removing their individual dimension lists.

DROP
USER G1
ARRAY ARRY→
DROP
USER G2
ARRAY ARRY→
DROP
USER G3
ARRAY ARRY→
DROP

```
3:             1.28E-12
2:                -0.89
1:                 0.45
→ARRY|ARRY→| PUT | GET |PUTI|GETI
```

Note the utility program →ROW, described in the section entitled "Orthogonality," could also be used to form the list of vectors above.

Next, key in the dimensions of the matrix that will be formed by the three vectors.

{ 3 3 } ENTER

```
3:                -0.89
2:                 0.45
1:        { 3.00 3.00 }
→ARRY|ARRY→| PUT | GET |PUTI|GETI
```

Finally, place the three vectors into matrix form.

→ARRY

```
1: [[ 0.41 0.41 0.82 ]
    [ 0.91 -0.18 -0.37…
    [ 1.28E-12 -0.89 0…
→ARRY|ARRY→| PUT | GET |PUTI|GETI
```

Compute the determinant.

DET

The determinant is −1. The matrix is non-singular, and the vectors form an orthonormal basis.

Purge the vectors $X_1, X_2, X_3, Y_1, Y_2, Y_3, G_1, G_2, G_3$ and, if desired, program NORM.

```
{'X1''X2''X3''Y1''Y2''Y3''G1''G2''G3'
 'NORM' PURGE
```


Eigenvalues

Another fundamental use for matrices is in developing a structure to represent linear transformations within a geometric system. Any matrix that represents a particular linear transformation reflects the properties of that transformation.

Since similar matrices share all the intrinsic geometric properties of a transformation, an important problem is to find a simple canonical form for each similarity class. This simple canonical form can be found by computing the eigenvalues and eigenvectors. Two methods for computing eigenvalues are illustrated, along with a method for finding eigenvectors.

The Characteristic Polynomial

The characteristic equation for a matrix can be written as

$$
\begin{aligned}
AX &= \lambda X \\
AX - \lambda X &= 0 \\
(A - \lambda I)X &= 0 \\
X &= 0 \quad \text{Trivial Solution} \\
\det(A - \lambda I) &= 0 \quad \text{Non-trivial Solution}
\end{aligned}
$$

Expansion of the non-trivial characteristic equation yields the characteristic polynomial

$$s_0\lambda^n + s_1\lambda^{n-1} + \cdots + s_{n-1}\lambda + s_n = 0$$

The three programs below combine to determine the characteristic polynomial for an arbitrary matrix on the stack.

The first program, TRCN, creates a list of the traces of the first n powers of the matrix.

The second program, SYM, uses the list created by TRCN to compute the coefficients of the characteristic polynomial.

The final program, PSERS, uses the coefficients from SYM and a variable name entered into level one to create an expression of the characteristic polynomial. Key in the first program.

```
[CLEAR]
« DUP SIZE 1 GET → g
n « g 'tmp' STO {} 1 n
START 0 1 n FOR i tmp i
DUP 2 →LIST GET + NEXT 1
→LIST + 'tmp' g STO*
NEXT 'tmp'
PURGE » » [ENTER] [<>]
```

```
1: « DUP SIZE 1.00 GET
   → g n « g 'tmp' STO
   { } 1.00 n START
   0.00 1.00 n FOR i
```

Store the program.

```
'TRCN [STO]
```

```
4:
3:
2:
1:
```


Key in the second program.

```
« DUP SIZE → b n «
{1} 1 n FOR i → s
« 0 1 i FOR j b j
GET s i j - 1 + GET ×
- NEXT i ÷ 1 →LIST s
SWAP + » NEXT » »
[ENTER] [<>]
```

```
1: « DUP SIZE → b n « {
   1.00 } 1.00 n FOR i
   → s « 0.00 1.00 i
   FOR j b j GET s i j
```

Store the program.

```
'SYM [STO]
```

```
4:
3:
2:
1:
```

Key in the final program.

```
« → x « LIST→ 0 SWAP
1 FOR n n 1 + ROLL
x n 1 - ^ × + -1
STEP » » [ENTER] [<>]
```

```
1: « → x « LIST→ 0.00
   SWAP 1.00 FOR n n
   1.00 + ROLL x n 1.00
   - ^ * + -1.00 STEP »
```

Store the program.

```
'PSERS [STO]
```

```
4:
3:
2:
1:
```

Find the characteristic polynomial for the following matrix.

$$ARR = \begin{bmatrix} -17 & -57 & -69 \\ 1 & 5 & 3 \\ 5 & 15 & 21 \end{bmatrix}$$

Key in the coefficient matrix.

`[[-17 -57 -69[1 5 3`
`[5 15 21` [ENTER]

```
2:
1: [[ -17.00 -57.00 -6...
    [ 1.00 5.00 3.00 ]
    [ 5.00 15.00 21.00...
```

Create a list of the traces of the first n powers for the matrix.

[USER] [TRCN]

```
2:
1: { 9.00 41.00 225.00
   }
PSERS SYM TRCN UT
```

Compute the coefficients of the characteristic polynomial.

[SYM]

```
2:
1: { 1.00 -9.00 20.00
   -12.00 }
PSERS SYM TRCN UT
```

Create the algebraic expression of the characteristic polynomial with the variable name L.

`'L'` [PSERS]

```
3:
2:
1:    'L^3-9*L^2+20*L-12'
PSERS SYM TRCN UT
```

The characteristic polynomial is

$$\lambda^3 - 9\lambda^2 + 20\lambda - 12$$

Store the polynomial as the current expression in EQ for the following problem section.

[SOLV]
[STEQ]

```
3:
2:
1:
STEQ RCEQ SOLVR ISOL QUAD SHOW
```


Compute Eigenvalues From Expansion

The eigenvalues of a matrix can be found by solving for the roots of the characteristic polynomial.

Find the eigenvalues for the characteristic polynomial stored as the current equation, EQ, in the previous problem section.

Clear the stack and set the display mode to two decimal places.

[CLEAR]
[MODE] 2 [FIX]

```
3:
2:
1:
STD FIX SCI ENG DEG RAD
```

Clear the current plot parameters.

[PLOT]
`'PPAR` [PURGE]

```
3:
2:
1:
PPAR RES AXES CENTR *W *H
```

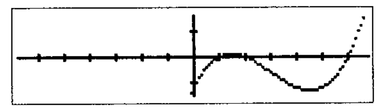
Adjust the plot height by ten.

`10` [*H]

```
3:
2:
1:
PPAR RES AXES CENTR *W *H
```

Draw a plot of the characteristic polynomial, which was stored in EQ in the previous problem.

[DRAW]



Note the three roots of the quadratic indicate three distinct eigenvalues for the 3×3 matrix ARR.

Use the solver to set guesses for the roots and solve for the three eigenvalues.

[ATTN]
[SOLV]
[SOLVR]

```
3:
2:
1:
L EXPR=
```

Make an initial guess of 0.5 for the first root.

0.5 [L]

```
L: 0.50
2:
1:
L  EXPR=
```

Solve for the first root.

To solve for a [SOLVR] variable, press the shift key followed by the desired [SOLVR] variable key. Pressing the [ENTER] key will display the intermediate values during calculation.

[ ] [L] [ENTER]

```
L: 1.00
Zero
1:                    1.00
L  EXPR=
```

The first eigenvalue $\lambda_1 = 1$.

Make an initial guess of 2.5 for the second eigenvalue.

[CLEAR]
2.5 [L]

```
L: 2.50
2:
1:
L  EXPR=
```

Solve for the root.

[ ] [L] [ENTER]

```
L: 2.00
Zero
1:                    2.00
L  EXPR=
```

The second eigenvalue $\lambda_2 = 2$.

Make an initial guess of 5 for the third eigenvalue.

[CLEAR]
5 [L]

```
L: 5.00
2:
1:
L  EXPR=
```

Solve for the root.

[ ] [L] [ENTER]

```
L: 6.00
Zero
1:                    6.00
L  EXPR=
```

The third eigenvalue $\lambda_3 = 6$.

Compute Eigenvectors

We can compute the eigenvectors corresponding to the three eigenvalues found in the previous problem.

$$ARR = \begin{bmatrix} -17 & -57 & -69 \\ 1 & 5 & 3 \\ 5 & 15 & 21 \end{bmatrix}$$

Clear the stack and set the display mode to one decimal place.

[CLEAR]
[MODE] 1 [FIX]

```
3:
2:
1:
STO FIX SCI ENG DEG RAD
```

Key in the matrix ARR.

```
[[-17 -57 -69[1 5 3
 [5 15 21 ENTER
```

```
1: [[ -17.0 -57.0 -69.…
    [ 1.0 5.0 3.0 ]
    [ 5.0 15.0 21.0 ]]
STO FIX SCI ENG DEG RAD
```

Create the 3 × 3 Identity matrix I.

3 [ENTER]

```
3:
2: [[ -17.0 -57.0 -69.…
1:                   3.0
STO FIX SCI ENG DEG RAD
```

[ARRAY] [IDN]

```
1: [[ 1.0 0.0 0.0 ]
    [ 0.0 1.0 0.0 ]
    [ 0.0 0.0 1.0 ]]
SIZE RDM TRN CON IDN RSD
```

Form $\lambda * I$ for $\lambda_1 = 1$.

1 [ENTER]
[×]

```
1: [[ 1.0 0.0 0.0 ]
    [ 0.0 1.0 0.0 ]
    [ 0.0 0.0 1.0 ]]
SIZE RDM TRN CON IDN RSD
```

Subtract from ARR to obtain the matrix $(ARR - \lambda_1 I)$.

[−]

```
1: [[ -18.0 -57.0 -69.…
    [ 1.0 4.0 3.0 ]
    [ 5.0 15.0 20.0 ]]
SIZE RDM TRN CON IDN RSD
```

Store the matrix $(ARR - \lambda_1 I)$ as *EIG*.

'EIG [STO]

```
3:
2:
1:
SIZE | RDM | TRN | CON | IDN | RSD
```

Recall the matrix *EIG*.

[USER] ≣EIG≣

```
1: [[ -18.0 -57.0 -69....
    [ 1.0 4.0 3.0 ]
    [ 5.0 15.0 20.0 ]]
EIG | L | PPAR | EQ | UT |
```

Verify that $\det(A - \lambda I) = 0$.

[ARRAY]
≣DET≣

```
3:
2:
1:                    0.0
CROSS | DOT | DET | ABS | RNRM | CNRM
```

The determinant is approximately zero.

Recall matrix *EIG* once more.

[DROP]
[USER] ≣EIG≣

```
1: [[ -18.0 -57.0 -69....
    [ 1.0 4.0 3.0 ]
    [ 5.0 15.0 20.0 ]]
EIG | L | PPAR | EQ | UT |
```

Reduce to row echelon form to solve for the eigenvector X_1, where $(A - \lambda_1 I)X_1 = 0$.

Enter [EDIT] mode and use the [DEL] key to remove the outer array brackets and form three individual row vectors. Each row vector corresponds to one equation of the system. After the edit, [ENTER] the row vectors.

Note that the utilities in the section entitled "Orthogonality" can also perform the modification of the form of the matrix.

```
[ -18 -57 -69 ]
 [ 1 4 3 ]
 [ 5 15 20 ] [ENTER]
```

```
3: [ -18.0 -57.0 -69.0...
2:         [ 1.0 4.0 3.0 ]
1:      [ 5.0 15.0 20.0 ]
EIG | L | PPAR | EQ | UT |
```

Use the program UT, described in the problem section "Homogeneous System," to reduce the matrix to upper triangular form.

≣UT≣

```
3: [ -18.0 -57.0 -69.0...
2:       [ 0.0 0.8 -0.8 ]
1: [ 0.0 0.0 -1.0E-11 ]
EIG | L | PPAR | EQ | UT |
```


Remove the vector that represents the equation 0 = 0.

[DROP]

```
3:
2: [ -18.0 -57.0 -69.0...
1:       [ 0.0 0.8 -0.8 ]
EIG | L | PPAR | EQ | UT |
```

Enter the equation represented by the second vector.

'.8 × X2 − .8 × X3 =0
[ENTER]

```
3: [ -18.0 -57.0 -69.0...
2:       [ 0.0 0.8 -0.8 ]
1:      '0.8*X2-0.8*X3=0'
EIG | L | PPAR | EQ | UT |
```

Solve for x_2.

[ALGEBRA]
'X2 [ENTER]

```
3:       [ 0.0 0.8 -0.8 ]
2:      '0.8*X2-0.8*X3=0'
1:                   'X2'
COLCT | EXPAN | SIZE | FORM | OBSUB | EXSUB
```

Isolate the term.

≣ISOL≣

```
3: [ -18.0 -57.0 -69.0...
2:       [ 0.0 0.8 -0.8 ]
1:          '0.8*X3/0.8'
TAYLR | ISOL | QUAD | SHOW | OBGET | EXGET
```

Collect terms.

≣COLCT≣

```
3: [ -18.0 -57.0 -69.0...
2:       [ 0.0 0.8 -0.8 ]
1:                   'X3'
COLCT | EXPAN | SIZE | FORM | OBSUB | EXSUB
```

The result is $x_2 = x_3$. Remove this solution and the second vector from the stack.

[DROP]
[DROP]

```
3:
2:
1: [ -18.0 -57.0 -69.0...
COLCT | EXPAN | SIZE | FORM | OBSUB | EXSUB
```

Enter the equation represented by the first vector, substituting x_3 with x_2.

'−18 × X1 −57 × X2
−69 × X2 =0 [ENTER]

```
2: [ -18.0 -57.0 -69.0...
1: '-18*X1-57*X2-69*X2=
   0'
COLCT | EXPAN | SIZE | FORM | OBSUB | EXSUB
```

Solve for x_1.

'X1 [ENTER]

```
3: [ -18.0 -57.0 -69.0…
2: '-18*X1-57*X2-69*X2…
1:                 'X1'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Isolate the term.

ISOL

```
2: [ -18.0 -57.0 -69.0…
1: '(69*X2+57*X2)/(-18)
   '
TAYLR ISOL QUAD SHOW OBGET EXGET
```

Collect like terms.

COLCT

```
3:
2: [ -18.0 -57.0 -69.0…
1:            '-(7.0*X2)'
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

The result is $x_1 = -7*x_2$.

Therefore a solution eigenvector is $x_1 = -7, x_2 = 1, x_3 = 1$, or $X_1 = [-7\ 1\ 1]$. Verify that $(A - \lambda I)X = 0$.

[CLEAR]
[-7 1 1 [ENTER]

```
3:
2:
1:      [ -7.0 1.0 1.0 ]
COLCT EXPAN SIZE FORM OBSUB EXSUB
```

Recall $(A - \lambda I)$.

[USER]
EIG

```
1: [[ -18.0 -57.0 -69.…
    [ 1.0 4.0 3.0 ]
    [ 5.0 15.0 20.0 ]]
EIG L PPAR EQ UT
```

Multiply the two matrices.

[SWAP]
[×]

```
3:
2:
1:       [ 0.0 0.0 0.0 ]
EIG L PPAR EQ UT
```


The result is 0, verifying that X_1 is indeed an eigenvector associated with λ_1.

The same procedure can be followed to find eigenvectors for $\lambda_2 = 2$ and $\lambda_3 = 6$.

Purge the user variables and programs used in the last three sections.

{'EIG''L''PPAR''EQ''UT' [PURGE]

Compute Eigenvalues from |λI − A|

Find eigenvalues directly from the function det($\lambda I - A$) without computing the characteristic polynomial.

$$A = \begin{bmatrix} -7.8 & -29.7 & -39.6 \\ 0 & 2.1 & 0 \\ 3.3 & 9.9 & 15.3 \end{bmatrix}$$

Clear the stack and set the display mode to two decimal places.

CLEAR
MODE 2 FIX

```
3:
2:
1:
STD  FIX■  SCI  ENG  DEG  RAD■
```

Clear the current plot parameters.

'PPAR PURGE

```
3:
2:
1:
STD  FIX■  SCI  ENG  DEG  RAD■
```

Key in the 3 × 3 matrix.

```
[[-7.8 -29.7 -39.6
[0 2.1 0[3.3 9.9 15.3
```
ENTER

```
1: [[ -7.80 -29.70 -39...
    [ 0.00 2.10 0.00 ]
    [ 3.30 9.90 15.30 ...
STD  FIX■  SCI  ENG  DEG  RAD■
```

Store matrix A.

'A STO

```
3:
2:
1:
STD  FIX■  SCI  ENG  DEG  RAD■
```

Enter a program that computes the function det($\lambda I - A$), with λ the independent variable.

« 3 IDN L × A − DET »
ENTER

```
2:
1: « 3.00 IDN L * A -
   DET »
STD  FIX■  SCI  ENG  DEG  RAD■
```

Store the function as the current expression in EQ.

PLOT
STEQ

```
3:
2:
1:
STEQ  RCEQ  PMIN  PMAX  INDEP  DRAW
```


Adjust the plot height.

5 *H

```
3:
2:
1:
PPAR  RES  AXES  CENTR  *W  *H
```

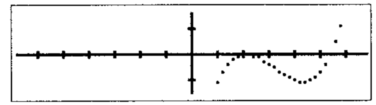
Set a larger resolution.

2 RES

```
3:
2:
1:
PPAR  RES  AXES  CENTR  *W  *H
```

Plot the function, using λ for the abscissa. The program takes several minutes to complete, as it computes the determinant for each point plotted.

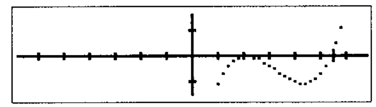
DRAW



The curve shows that there are only two distinct roots. The leftmost root, which is a local maximum, must represent a double eigenvalue root.

Digitize the roots to set initial guesses for the root solver.

\> . . . > INS
\> . . . > INS



Set the standard display mode.

ATTN
MODE STD

```
3:
2:                (2.1,0)
1:                (5.3,0)
STD■  FIX  SCI  ENG  DEG  RAD■
```

Note: The values displayed will vary by differences in the digitizing position from the graphics display.

Use the Solver to find the roots of the curve.

SOLV
SOLVR

```
3:
2:                (2.1,0)
1:                (5.3,0)
L  A  EXPR=
```

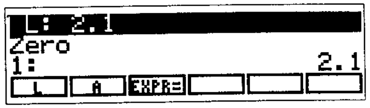
Solve for the rightmost root.

[L] [ ] [L]

One root is $\lambda_1 = 5.40$.

Drop this result from the stack and solve for the next root.

[DROP]
[L] [ ] [L]



The double eigenvalue is $\lambda_2 = \lambda_3 = 2.10$.

Least Squares

The method of least squares is a standard statistical algorithm used to fit a curve to data in order to estimate a function, predict a trend, or approximate missing data values. Least squares results can easily be calculated on the HP-28S or HP-28C, and the graphic display is particularly useful for examining the fit to the original data.

Straight Line Fitting

Find the least squares straight line fit to the four points: (0,1), (1,3), (2,4), and (3,4).

The least squares solution is given by $Y = MV$ to fit the line $y = ax + b$.

Note: The solution provided below serves to illustrate matrix operations, and could be replaced, in the case of $y = ax + b$, with the statistical functions (Linear Regression) of the HP-28S or HP-28C.

$$Y = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solving for V gives

$$V = \frac{M^T Y}{M^T M}$$

CLEAR
MODE 2 FIX

```
3:
2:
1:
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the y values of the data points.

[[1[3[4[4 ENTER

```
1: [[ 1.00 ]
    [ 3.00 ]
    [ 4.00 ]
STO | FIX■ | SCI | ENG | DEG | RAD■
```


Store the 4 × 1 matrix Y.

'Y STO

```
3:
2:
1:
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the a and b values representing the line $y = ax + b$.

[[0 1[1 1[2 1[3 1 ENTER

```
1: [[ 0.00 1.00 ]
    [ 1.00 1.00 ]
    [ 2.00 1.00 ]
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Store the 4 × 2 matrix M.

'M STO

```
3:
2:
1:
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Compute V using the least squares fitting method.

M ENTER
ARRAY
TRN
Y ×

```
2:
1: [[ 23.00 ]
    [ 12.00 ]]
SIZE | RDM | TRN | CON | IDN | RSD
```

M ENTER
TRN
M ×

```
2: [[ 23.00 ]  [ 12.00…
1: [[ 14.00 6.00 ]
    [ 6.00 4.00 ]]
SIZE | RDM | TRN | CON | IDN | RSD
```

÷

```
2:
1: [[ 1.00 ]
    [ 1.50 ]]
SIZE | RDM | TRN | CON | IDN | RSD
```

Store the coefficients from matrix V in the individual variables a and b.

ARRY→

```
3:                1.00
2:                1.50
1:         { 2.00 1.00 }
→ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Drop the dimension list.

DROP

```
3:
2:                1.00
1:                1.50
→ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Store the two coefficients.

'B [STO]

```
3:
2:
1:                    1.00
→ARRY ARRY→ PUT GET PUTI GETI
```

'A [STO]

```
3:
2:
1:
→ARRY ARRY→ PUT GET PUTI GETI
```

Enter the equation for the straight line.

'A × X + B [ENTER]

```
3:
2:
1:                 'A*X+B'
→ARRY ARRY→ PUT GET PUTI GETI
```

Store the equation.

'LINE [STO]

```
3:
2:
1:
→ARRY ARRY→ PUT GET PUTI GETI
```

Recall equation LINE.

[USER] [LINE]

```
3:
2:
1:                 'A*X+B'
LINE A B M Y
```

Store the line equation as the current expression in EQ.

[SOLV] [STEQ]

```
3:
2:
1:
STEQ RCEQ SOLVR ISOL QUAD SHOW
```

Use the Solver to compute the desired line.

[SOLVR] [EXPR=]

```
EXPR='1.00*X+1.50'
2:
1:           '1.00*X+1.50'
A X B EXPR=
```

The straight line fit to the data is the equation $y = x + 1.5$.

Now use the PLOT menu to draw the line and verify the fit to the data.

Clear the current plot parameters.

[PLOT]
'PPAR [PURGE]

```
3:
2:
1:           '1.00*X+1.50'
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Establish X as the independent variable.

'X [INDEP]

```
3:
2:
1:           '1.00*X+1.50'
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Adjust the height by 5.

5 [*H]

```
3:
2:
1:           '1.00*X+1.50'
PPAR RES AXES CENTR *W *H
```

Recenter the axes so that the point (0,1) can be viewed on the plot.

(-1,-1) [ENTER]
[AXES]

```
3:
2:
1:           '1.00*X+1.50'
PPAR RES AXES CENTR *W *H
```

Now move to the Statistics menu to set up a scatter plot.

[STAT] [CLΣ]

```
3:
2:
1:           '1.00*X+1.50'
Σ+ Σ- NΣ CLΣ STOΣ RCLΣ
```

Enter the four data points into ΣDAT.

[0,1 [Σ+]
[1,3 [Σ+]
[2,4 [Σ+]
[3,4 [Σ+]

```
3:
2:
1:           '1.00*X+1.50'
Σ+ Σ- NΣ CLΣ STOΣ RCLΣ
```

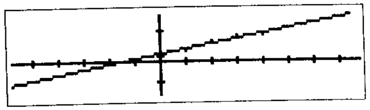
Enter a program that will overlay the function plot onto the scatter plot.

[PLOT]
« CLLCD DRWΣ DRAW
[ENTER]

```
3:
2:           '1.00*X+1.50'
1:   « CLLCD DRWΣ DRAW »
STEQ RCEQ PMIN PMAX INDEP DRAW
```

Draw the plot.

EVAL



We can see from the plot that the line fits the four points well.

Purge the variables used in the problem section.

```
{'ΣPAR''ΣDAT''PPAR''EQ''A''B''M''Y''TIME'
```

PURGE

Quadratic Polynomial

According to Newton's Second Law of Motion, a body near the earth's surface falls vertically downward according to the equation

$$y = y_0 + v_0 t + \frac{1}{2} g t^2$$

where

y = vertical displacement at time t.
y_0 = initial vertical displacement at time $t_0 = 0$.
v_0 = initial velocity at time $t_0 = 0$.
g = Newton's constant of acceleration of gravity near the earth's surface.

An experiment is performed to evaluate g. A weight is released with unknown initial displacement and velocity. At a fixed time interval the distance fallen from a fixed reference point is measured. The following results are obtained: At times t = .1, .2, ...5 seconds the weight has fallen y = −.055, .094, .314, .756, and 1.138 meters, respectively, from the reference point. Calculate the value for Newton's constant g using these data.

We will fit the quadratic curve

$$y = a + bt + ct^2$$

to the five data points. The least squares solution is given by

$$Y = MV$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{bmatrix}$$

and

$$V = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Solving for V gives

$$V = \frac{M^T Y}{M^T M}$$

Clear the stack and set the display mode to three decimal places.

[CLEAR]
[MODE] 3 [FIX]

```
3:
2:
1:
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the matrix of y values.

`[[-.055[.094[.314[.756`
`[1.138` [ENTER]

```
1: [[ -0.055 ]
    [ 0.094 ]
    [ 0.314 ]
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Store the 5×1 matrix Y.

`'Y` [STO]

```
3:
2:
1:
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the components of array M.

Enter $row_1 = 1, .1, .1^2$.

`1` [ENTER]
`.1` [ENTER]
[ENTER]
[x^2]

```
3:                1.000
2:                0.100
1:                0.010
STO | FIX■ | SCI | ENG | DEG | RAD■
```


Enter $row_2 = 1, .2, .2^2$.

`1` [ENTER]
`.2` [ENTER]
[ENTER]
[x^2]

```
3:                1.000
2:                0.200
1:                0.040
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Enter $row_3 = 1, .3, .3^2$.

`1` [ENTER]
`.3` [ENTER]
[ENTER]
[x^2]

```
3:                1.000
2:                0.300
1:                0.090
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Enter $row_4 = 1, .4, .4^2$.

`1` [ENTER]
`.4` [ENTER]
[ENTER]
[x^2]

```
3:                1.000
2:                0.400
1:                0.160
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Finally, enter $row_5 = 1, .5, .5^2$.

`1` [ENTER]
`.5` [ENTER]
[ENTER]
[x^2]

```
3:                1.000
2:                0.500
1:                0.250
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Key in the dimension of M.

`{5 3` [ENTER]

```
3:                0.500
2:                0.250
1:        { 5.000 3.000 }
STO | FIX■ | SCI | ENG | DEG | RAD■
```

Put the components into the array.

[ARRAY]
[→ARRY]

```
1: [[ 1.000 0.100 0.01…
    [ 1.000 0.200 0.04…
    [ 1.000 0.300 0.09…
→ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Store matrix M.

`'M` [STO]

```
3:
2:
1:
→ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Compute V using the least squares method.

M [ENTER]
[TRN]
Y [×]

```
1: [[ 2.247 ]
    [ 0.979 ]
    [ 0.437 ]]
 SIZE | RDM | TRN | CON | IDN | RSD
```

M [ENTER]
[TRN]
M [×]
[÷]

```
1: [[ -0.121 ]
    [ 0.099 ]
    [ 4.914 ]]
 SIZE | RDM | TRN | CON | IDN | RSD
```

Store matrix V.

'V [STO]

```
3:
2:
1:
 SIZE | RDM | TRN | CON | IDN | RSD
```

Evaluate g, Newton's constant of gravity. Get element c from the solution vector V, then multiply c by 2. $g = 2*c$.

V [ENTER]
{3 1}
[GET]
2 [×]

```
3:
2:
1:                  9.829
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Convert from m/sec^2 to ft/sec^2.

[LC] m [ENTER]
[LC] ft [ENTER]

```
3:                  9.829
2:                    'm'
1:                   'ft'
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

[CONVERT]

```
3:
2:                 32.246
1:                   'ft'
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

The result is $g = 32.246$ ft/sec^2.

Now use the solver to compute the desired quadratic polynomial.

'A+B×T+C×T^2
[ENTER]

```
3:                 32.246
2:                   'ft'
1:         'A+B*T+C*T^2'
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Store equation POLY.

'POLY [STO]

```
3:
2:                 32.246
1:                   'ft'
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Get the coefficients from matrix V.

V [ENTER]

```
1: [[ -0.121 ]
    [ 0.099 ]
    [ 4.914 ]]
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

[ARRY→]

```
3:                  0.099
2:                  4.914
1:        { 3.000 1.000 }
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Drop the dimension list.

[DROP]

```
3:                 -0.121
2:                  0.099
1:                  4.914
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Store the three coefficients *a*, *b*, and *c*.

'C [STO]

```
3:                   'ft'
2:                 -0.121
1:                  0.099
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

'B [STO]

```
3:                 32.246
2:                   'ft'
1:                 -0.121
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

'A [STO]

```
3:
2:                 32.246
1:                   'ft'
 →ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Recall the equation.

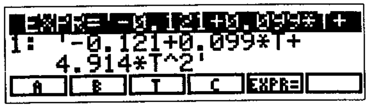
USER POLY

Store the equation as the current expression EQ.

SOLV
STEQ

Use the Solver to compute the desired equation.

SOLVR
EXPR=



The least squares solution equation is $-0.121+0.099t+4.914t^2$.

Next, overlay the function curve over a scatter plot of the data points to verify the fit.

First, clear the current plot parameters and establish *t* as the independent variable.

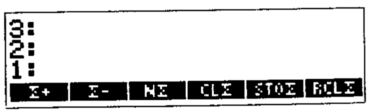
CLEAR
PLOT 'PPAR PURGE
'T INDEP

Adjust the plot width by .1, to plot 0.1 second intervals along the abscissa.

.1 *W

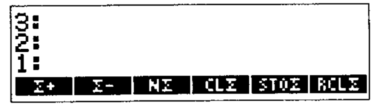
Next use the Statistics menu to create the scatter plot.

STAT
CLΣ



Enter the data points for the scatter plot.

[.1 -.055 Σ+
[.2 .094 Σ+
[.3 .314 Σ+
[.4 .756 Σ+
[.5 1.138 Σ+



Now write a program to overlay the two plots.

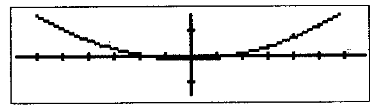
PLOT
« CLLCD DRWΣ DRAW
ENTER

Store program PLT.

'PLT STO

Draw the plot.

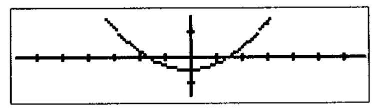
USER PLT



You may wish to rescale the plot height to obtain a better view of the fit of the first two data points.

PLOT
.25 *H

USER
PLT



The plots show a good fit of the quadratic polynomial to the five data points.

Purge the user variables and programs created in this example.

{'ΣPAR''PLT''ΣDAT''PPAR''EQ''A''B''C''POLY'
'V''M''Y' PURGE

Markov Chains

A Markov Chain is a system that moves from state to state, and in which the probability of transition to a next state depends only on the preceding state. The system states can be predicted at particular points in time using transition probabilities.

The transition matrix for the Markov Process is the $n \times n$ matrix $P = [p_{ij}]$ where p_{ij} = probability of transition directly from state j to state i, and $\sum_{i=1}^{n} p_{ij} = 1$.

The components of the state vector $X^{(n)}$ signify the probability that the system is in state i at the n^{th} observation.

$$X^{(n)} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

The model for the system is described by $X^{(n+1)} = P\,X^{(n)}$, where the transition matrix applied to the current state determines the next state.

Steady State of a System

A chemist runs an experiment where colored films are immersed in a solution for a brief time period, resulting in a possible color change. She calculates the color changes according to the following probabilities.

Original Color			New Color
Magenta	Cyan	Yellow	
.8	.3	.2	Magenta
.1	.2	.6	Cyan
.1	.5	.2	Yellow

Determine to two decimal places the probable future color of a cyan film dipped in the solution several times.

CLEAR
MODE 2 FIX

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Key in the 3 × 3 transition matrix P.

`[[.8 .3 .2[.1 .2 .6[.1 .5 .2` ENTER

```
1: [[ 0.80 0.30 0.20 ]
    [ 0.10 0.20 0.60 ]
    [ 0.10 0.50 0.20 ]]
STO FIX■ SCI ENG DEG RAD■
```

`'P` STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Key in the initial state vector X^0. This vector represent an initial state of cyan.

`[[0[1[0` ENTER

```
1: [[ 0.00 ]
    [ 1.00 ]
    [ 0.00 ]]
STO FIX■ SCI ENG DEG RAD■
```

`'X` STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Key in the initial value for n = current state.

0 [ENTER]
'N [STO]

```
3:
2:
1:
STO | FIX | SCI | ENG | DEG | RAD
```

Write a program to compute the next future state.

« N 1 + 'N' STO P
SWAP × » [ENTER]

```
2:
1: « N 1.00 + 'N' STO P
   SWAP * »
STO | FIX | SCI | ENG | DEG | RAD
```

Store program MARK.

'MARK [STO]

```
3:
2:
1:
STO | FIX | SCI | ENG | DEG | RAD
```

Recall the initial state vector.

[USER] [X]

```
1: [[ 0.00 ]
    [ 1.00 ]
    [ 0.00 ]]
MARK | N | X | P | |
```

Compute the next state.

[MARK]

```
1: [[ 0.30 ]
    [ 0.20 ]
    [ 0.50 ]]
MARK | N | X | P | |
```

After one observation, the color is most likely to be yellow. Compute the next state.

[MARK]

```
1: [[ 0.40 ]
    [ 0.37 ]
    [ 0.23 ]]
MARK | N | X | P | |
```

After two observations, the color is most likely to be either magenta or cyan. Continue computing future states until a final steady state is reached.

[MARK]

```
1: [[ 0.48 ]
    [ 0.25 ]
    [ 0.27 ]]
MARK | N | X | P | |
```


[MARK]

```
1: [[ 0.51 ]
    [ 0.26 ]
    [ 0.23 ]]
MARK | N | X | P | |
```

[MARK]

```
1: [[ 0.53 ]
    [ 0.24 ]
    [ 0.23 ]]
MARK | N | X | P | |
```

[MARK]

```
1: [[ 0.54 ]
    [ 0.24 ]
    [ 0.22 ]]
MARK | N | X | P | |
```

[MARK]

```
1: [[ 0.55 ]
    [ 0.23 ]
    [ 0.22 ]]
MARK | N | X | P | |
```

[MARK]

```
1: [[ 0.55 ]
    [ 0.23 ]
    [ 0.21 ]]
MARK | N | X | P | |
```

[MARK]

```
1: [[ 0.56 ]
    [ 0.23 ]
    [ 0.21 ]]
MARK | N | X | P | |
```

[MARK]

```
1: [[ 0.56 ]
    [ 0.23 ]
    [ 0.21 ]]
MARK | N | X | P | |
```

The system has reached a steady state. Determine how many observations were completed to reach this final state.

[N]

```
3:
2: [[ 0.56 ]  [ 0.23 ]…
1:                  10.00
MARK | N | X | P | |
```

The system reaches a steady state after $n = 10$ observations. The probable future color of an initially cyan film immersed several times is .56 magenta, .23 cyan, and .21 yellow.

Purge the variables used in this problem section.

{'MARK''N''X''P' [PURGE]

A Sample Application

Matrix manipulations are used to solve complex, multi-dimensional problems. The following applications illustrate use of the HP-28S or HP-28C matrix capabilities in a market with challenging economic issues. These same analytical tools can be applied in many other industries.

Forest Management

When a forest is managed by a sustainable harvesting policy, every tree harvested is replaced by a new seedling, so the total population quantity remains constant. A matrix model can be developed to assist in determining optimal harvesting procedures. The model is based on categorizing the trees into height/price classes and computing an optimal sustainable yield for a long-range time period.

The Sustainable Harvesting Cycle is represented by:

Forest ready for harvest − harvest + new seedlings = forest after harvest, or

$$GX - Y + RY = X$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

X = Nonharvest vector, the trees that remain after the harvest and replanting.

x_i = number of trees in the ith class.

i ranges from 1 to n, where there are n height/price classes.

$S = \sum_{i=1}^{n} x_i$ = total number of trees sustained.

Tree growth between harvests is designated by g_i, the fraction of trees that grow from class i to class $i+1$.

$1 - g_i$ = fraction of trees that remain in class i.

The growth matrix is

$$G = \begin{bmatrix} 1-g_1 & 0 & 0 & \cdot & 0 \\ g_1 & 1-g_2 & 0 & \cdot & 0 \\ 0 & g_2 & 1-g_3 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1-g_{n-1} & 0 \\ 0 & 0 & 0 & g_{n-1} & 1 \end{bmatrix}$$

GX = Nonharvest vector after growth period, or forest ready for harvest.

$$Y = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

Y = Harvest vector, or trees removed at harvest.

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R = Replacement matrix.

RY = New seedling vector, or trees planted after harvest.

The Harvest Model

A harvester has a crop of 120 silver fir trees to sell annually for Christmas trees. After last year's harvest, his forest had the following configuration.

Class (i)	Height Interval in Feet (h_i)	Number of Trees (x_i)
1	[0,4)	15
2	[4,8)	20
3	[8,12)	35
4	[12,16)	30
5	[16,∞)	20

During the growth period, six trees in class 1 grew to the next height class, as did thirteen trees in class 2, ten trees in class 3, and four trees in class 4. If the harvester sustainably harvests eight trees of class 2, six trees of class 3, thirteen trees of class 4, and six trees of class 5, what is the configuration of his crop after harvest and replanting?

CLEAR
MODE 2 FIX

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Enter the 5 × 1 nonharvest vector X.

[[15[20[35[30[20 ENTER

```
1: [[ 15.00 ]
    [ 20.00 ]
    [ 35.00 ]
STO FIX■ SCI ENG DEG RAD■
```

'X STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Compute the growth fractions for each height class. First, compute $g_1 = 6/x_1$.

6 ENTER
15 ÷

```
3:
2:
1:                    0.40
STO FIX■ SCI ENG DEG RAD■
```


'G1 STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Compute $g_2 = 13/x_2$.

13 ENTER
20 ÷

```
3:
2:
1:                    0.65
STO FIX■ SCI ENG DEG RAD■
```

'G2 STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Compute $g_3 = 10/x_3$.

10 ENTER
35 ÷

```
3:
2:
1:                    0.29
STO FIX■ SCI ENG DEG RAD■
```

'G3 STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Compute $g_4 = 4/x_4$.

4 ENTER
30 ÷

```
3:
2:
1:                    0.13
STO FIX■ SCI ENG DEG RAD■
```

'G4 STO

```
3:
2:
1:
STO FIX■ SCI ENG DEG RAD■
```

Enter the 5 × 5 growth matrix G.

Enter row$_1$.

USER
1 ENTER
G1
–
0 ENTER
ENTER
ENTER
ENTER

```
3:                 0.00
2:                 0.00
1:                 0.00
 G4 | G3 | G2 | G1 | % |
```

Enter row$_2$.

G1
1 ENTER
G2
–
0 ENTER
ENTER
ENTER

```
3:                 0.00
2:                 0.00
1:                 0.00
 G4 | G3 | G2 | G1 | % |
```

Enter row$_3$.

0 ENTER
G2
1 ENTER
G3
–
0 ENTER
ENTER

```
3:                 0.71
2:                 0.00
1:                 0.00
 G4 | G3 | G2 | G1 | % |
```

Enter row$_4$.

0 ENTER
ENTER
G3
1 ENTER
G4
–
0 ENTER

```
3:                 0.29
2:                 0.87
1:                 0.00
 G4 | G3 | G2 | G1 | % |
```


Enter row$_5$.

0 ENTER
ENTER
ENTER
G4
1 ENTER

```
3:                 0.00
2:                 0.13
1:                 1.00
 G4 | G3 | G2 | G1 | % |
```

Enter the dimensions of G.

{5 5} ENTER

```
3:                 0.13
2:                 1.00
1:        { 5.00 5.00 }
 G4 | G3 | G2 | G1 | % |
```

Store matrix G.

ARRAY
→ARRY

```
1: [[ 0.60 0.00 0.00 0…
    [ 0.40 0.35 0.00 0…
    [ 0.00 0.65 0.71 0…
→ARRY|ARRY→| PUT | GET | PUTI | GETI
```

'G STO

```
3:
2:
1:
→ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Enter the 5 × 1 harvest vector Y.

[[0[8[6[13[6 ENTER

```
1: [[ 0.00 ]
    [ 8.00 ]
    [ 6.00 ]
→ARRY|ARRY→| PUT | GET | PUTI | GETI
```

'Y STO

```
3:
2:
1:
→ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Create the replacement matrix R. First enter the dimensions of R.

{5 5} ENTER

```
3:
2:
1:        { 5.00 5.00 }
→ARRY|ARRY→| PUT | GET | PUTI | GETI
```

Create a constant matrix whose entries are all zero.

0 [ENTER]
[CON]

```
1: [[ 0.00 0.00 0.00 0...
    [ 0.00 0.00 0.00 0...
    [ 0.00 0.00 0.00 0...
SIZE RDM TRN CON IDN RSD
```

Now enter ones across the entire first row of R.

{1 1} [ENTER]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:        { 1.00 1.00 }
SIZE RDM TRN CON IDN RSD
```

1 [PUTI]
1 [PUTI]
1 [PUTI]
1 [PUTI]
1 [PUTI]

```
3:
2: [[ 1.00 1.00 1.00 1...
1:        { 2.00 1.00 }
→ARRY ARRY→ PUT GET PUTI GETI
```

Drop the index list.

[DROP]

```
1: [[ 1.00 1.00 1.00 1...
    [ 0.00 0.00 0.00 0...
    [ 0.00 0.00 0.00 0...
→ARRY ARRY→ PUT GET PUTI GETI
```

Store matrix R.

'R [STO]

```
3:
2:
1:
→ARRY ARRY→ PUT GET PUTI GETI
```

Write a program to compute the configuration of the forest after harvest.

[USER]
« G X × Y − R Y × + »
[ENTER]

```
2:
1: « G X * Y - R Y * +
   »
R Y G G4 G3 G2
```

'CROP [STO]

```
3:
2:
1:
CROP R Y G G4 G3
```


Compute the new nonharvest vector with program CROP.

[CROP]

```
1: [[ 42.00 ]
    [ 5.00 ]
    [ 32.00 ]
CROP R Y G G4 G3
```

Use [EDIT] or [VIEW↓] to view the entire vector. The [ATTN] key will exit EDIT mode.

The new nonharvest vector is

$$X = \begin{bmatrix} 42 \\ 5 \\ 32 \\ 23 \\ 18 \end{bmatrix}$$

The program can be used with the new nonharvest vector to predict new forest configurations using the same harvesting cycle annually.

HP-28C users should purge the following variables and programs before continuing to the next portion of this example:

{'CROP' 'R' 'G' 'X' 'Y' [PURGE]

It is not necessary to purge these programs and variables if you are using an HP-28S.

Optimal Yield

If the harvester wishes to optimize his profit year after year, he must determine the optimal sustainable yield. This is achieved by harvesting all of the trees from one particular height/price class and no trees from any other class. The sustainable yield is thus a function of both price and growth rate, but independent of the current nonharvest vector. Note that if class k provides the maximum yield, the first year all classes $\geq k$ are harvested. In the following years only class k is harvested, and no trees will ever be present in higher classes.

S = total number of trees sustained in the forest.

$$P = \begin{bmatrix} p_1 & 0 & \cdot & 0 \\ \cdot & p_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & p_n \end{bmatrix} = \text{Price matrix}$$

p_i = price attained for class i.

$$GG = \begin{bmatrix} gg_1 \\ gg_2 \\ \cdot \\ \cdot \\ gg_n \end{bmatrix}$$

GG = growth ratio matrix.

where

$$\begin{cases} gg_i = \dfrac{1}{\sum_{k=1}^{i-1} \dfrac{1}{g_k}} & for\ i=2...n \\ gg_1 = 0 \end{cases}$$

$$YL = \begin{bmatrix} yl_1 \\ yl_2 \\ \cdot \\ \cdot \\ yl_n \end{bmatrix}$$

YL = yield vector.

yl_k = yield (total dollar amount) obtained by harvesting all of class i and no other class.

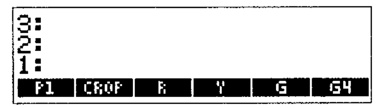
The optimal class to harvest can be selected by finding the maximum yl_k from yield vector YL, where

$$YL = P * S * GG$$

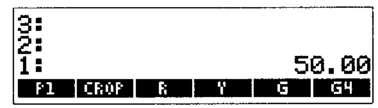
Suppose the market prices for the five classes are p_1 = \$0, p_2 = \$50, p_3 = \$100, p_4 = \$150, and p_5 = \$200. Determine which height class should be harvested.

Enter the market prices for the five classes and store in variables p_1 through p_5.

[CLEAR] [USER]
0 [ENTER]
'P1 [STO]

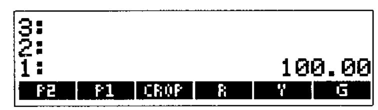


50 [ENTER]



'P2 [STO]

[P2]
2 [×]



'P3 [STO]

```
3:
2:
1:
 P3 | P2 | P1 | CROP | R | Y
```

[P2]
3 [×]

```
3:
2:
1:                150.00
 P3 | P2 | P1 | CROP | R | Y
```

'P4 [STO]

```
3:
2:
1:
 P4 | P3 | P2 | P1 | CROP | R
```

[P2]
4 [×]

```
3:
2:
1:                200.00
 P4 | P3 | P2 | P1 | CROP | R
```

'P5 [STO]

```
3:
2:
1:
 P5 | P4 | P3 | P2 | P1 | CROP
```

Enter the dimensions of P.

{5 5} [ENTER]

```
3:
2:
1:          { 5.00 5.00 }
 P5 | P4 | P3 | P2 | P1 | CROP
```

Create the 5×5 price matrix P. Since P is a sparse matrix, with most entries equal to zero, first create a constant array whose entries are all zero.

0 [ENTER]
[ARRAY] [CON]

```
1: [[ 0.00 0.00 0.00 0...
    [ 0.00 0.00 0.00 0...
    [ 0.00 0.00 0.00 0...
 SIZE | RDM | TRN | CON | IDN | RSD
```


Now enter the values p_i along the diagonal entries.

{1 1} [ENTER]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:          { 1.00 1.00 }
 SIZE | RDM | TRN | CON | IDN | RSD
```

P1 [ENTER]

```
3: [[ 0.00 0.00 0.00 0...
2:          { 1.00 1.00 }
1:                  0.00
 SIZE | RDM | TRN | CON | IDN | RSD
```

[PUTI]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:          { 1.00 2.00 }
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Use the [EDIT] function to modify the displayed position index. The modified position index is then [ENTER]ed. Alternatively, you may [DROP] {1.00 2.00} from above and enter the position index {2 2}.

[DROP] {2 2} [ENTER]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:          { 2.00 2.00 }
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

P2 [ENTER]

```
3: [[ 0.00 0.00 0.00 0...
2:          { 2.00 2.00 }
1:                 50.00
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

[PUTI]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:          { 2.00 3.00 }
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Use the [EDIT] function to modify the position index. The modified position index is then [ENTER]ed:

[DROP] {3 3} [ENTER]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:          { 3.00 3.00 }
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

P3 [ENTER]

```
3: [[ 0.00 0.00 0.00 0...
2:         { 3.00 3.00 }
1:                100.00
→ARRY ARRY→ PUT  GET  PUTI GETI
```

[PUTI]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:         { 3.00 4.00 }
→ARRY ARRY→ PUT  GET  PUTI GETI
```

Use the [EDIT] function to modify the position index. The modified position index is then [ENTER]ed:

[DROP] { 4 4 } [ENTER]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:         { 4.00 4.00 }
→ARRY ARRY→ PUT  GET  PUTI GETI
```

P4 [ENTER]

```
3: [[ 0.00 0.00 0.00 0...
2:         { 4.00 4.00 }
1:                150.00
→ARRY ARRY→ PUT  GET  PUTI GETI
```

[PUTI]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:         { 4.00 5.00 }
→ARRY ARRY→ PUT  GET  PUTI GETI
```

Use the [EDIT] function to modify the position index. The modified position index is then [ENTER]ed:

[DROP] { 5 5 } [ENTER]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:         { 5.00 5.00 }
→ARRY ARRY→ PUT  GET  PUTI GETI
```

P5 [ENTER]

```
3: [[ 0.00 0.00 0.00 0...
2:         { 5.00 5.00 }
1:                200.00
→ARRY ARRY→ PUT  GET  PUTI GETI
```

[PUTI]

```
3:
2: [[ 0.00 0.00 0.00 0...
1:         { 1.00 1.00 }
→ARRY ARRY→ PUT  GET  PUTI GETI
```


Drop the index string.

[DROP]

```
1: [[ 0.00 0.00 0.00 0...
    [ 0.00 50.00 0.00 ...
    [ 0.00 0.00 100.00...
→ARRY ARRY→ PUT  GET  PUTI GETI
```

Store matrix P.

'P [STO]

```
3:
2:
1:
→ARRY ARRY→ PUT  GET  PUTI GETI
```

Store the total number of trees sustained in variable S.

120 [ENTER]

```
3:
2:
1:                120.00
→ARRY ARRY→ PUT  GET  PUTI GETI
```

'S [STO]

```
3:
2:
1:
→ARRY ARRY→ PUT  GET  PUTI GETI
```

Compute the 5×1 growth ratio matrix GG.

Enter $gg_1 = 0$.

0 [ENTER]
'GG1 [STO]

```
3:
2:
1:
→ARRY ARRY→ PUT  GET  PUTI GETI
```

Compute $gg_2 = 1/g_1$.

[USER] [G1]
[1/x]

```
3:
2:
1:                  2.50
GG1  G4   G3   G2   G1   S
```

'GG2 [STO]

```
3:
2:
1:
GG2  GG1  G4   G3   G2   G1
```

Compute $gg_3 = 1/g_1 + 1/g_2$.

[GG2]
[G2]
[1/x]

```
3:
2:                  2.50
1:                  1.54
GG2  GG1  G4   G3   G2   G1
```

Keys	Display
[+]	3: 2: 1: 4.04 GG2 | GG1 | G4 | G3 | G2 | G1
'GG3 [STO]	3: 2: 1: GG3 | GG2 | GG1 | G4 | G3 | G2

Compute $gg_4 = 1/g_1 + 1/g_2 + 1/g_3$.

Keys	Display
GG3 G3 [1/x]	3: 2: 4.04 1: 3.50 GG3 | GG2 | GG1 | G4 | G3 | G2
[+]	3: 2: 1: 7.54 GG3 | GG2 | GG1 | G4 | G3 | G2
'GG4 [STO]	3: 2: 1: GG4 | GG3 | GG2 | GG1 | G4 | G3

Compute $gg_5 = 1/g_1 + 1/g_2 + 1/g_3 + 1/g_4$.

Keys	Display
GG4 G4 [1/x]	3: 2: 7.54 1: 7.50 GG4 | GG3 | GG2 | GG1 | G4 | G3
[+]	3: 2: 1: 15.04 GG4 | GG3 | GG2 | GG1 | G4 | G3
'GG5 [STO]	3: 2: 1: GG5 | GG4 | GG3 | GG2 | GG1 | G4

Now invert gg_2, gg_3, gg_4, and gg_5 to form the actual entries into matrix GG.

Keys	Display
GG2 [1/x]	3: 2: 1: 0.40 GG5 | GG4 | GG3 | GG2 | GG1 | G4

Keys	Display
'GG2 [STO]	3: 2: 1: GG5 | GG4 | GG3 | GG2 | GG1 | G4
GG3 [1/x]	3: 2: 1: 0.25 GG5 | GG4 | GG3 | GG2 | GG1 | G4
'GG3 [STO]	3: 2: 1: GG5 | GG4 | GG3 | GG2 | GG1 | G4
GG4 [1/x]	3: 2: 1: 0.13 GG5 | GG4 | GG3 | GG2 | GG1 | G4
'GG4 [STO]	3: 2: 1: GG5 | GG4 | GG3 | GG2 | GG1 | G4
GG5 [1/x]	3: 2: 1: 0.07 GG5 | GG4 | GG3 | GG2 | GG1 | G4
'GG5 [STO]	3: 2: 1: GG5 | GG4 | GG3 | GG2 | GG1 | G4

Create the 5×1 matrix GG. Put the elements on the stack.

Keys	Display
GG1 GG2 GG3 GG4 GG5	3: 0.25 2: 0.13 1: 0.07 GG5 | GG4 | GG3 | GG2 | GG1 | G4

Enter the matrix dimensions.

{5 1 [ENTER]

```
3:                    0.13
2:                    0.07
1:          { 5.00 1.00 }
 GG5 | GG4 | GG3 | GG2 | GG1 | G4
```

Create the matrix.

[ARRAY]
[→ARRY]

```
1: [[ 0.00 ]
    [ 0.40 ]
    [ 0.25 ]
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Store matrix *GG*.

'GG [STO]

```
3:
2:
1:
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Write a program to compute the yield vector.

« S P × GG × » [ENTER]

```
3:
2:
1:           « S P * GG * »
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Store program YLD.

'YLD [STO]

```
3:
2:
1:
 →ARRY | ARRY→ | PUT | GET | PUTI | GETI
```

Compute the 5×1 yield vector *YL*.

[USER]
[YLD]

```
1: [[ 0.00 ]
    [ 2400.00 ]
    [ 2971.43 ]
 YLD | GG | GG5 | GG4 | GG3 | GG2
```

You can use [EDIT] or [VIEW↓] to view the entire vector.

$$YL = \begin{bmatrix} 0 \\ 2400.00 \\ 2971.43 \\ 2387.76 \\ 1595.91 \end{bmatrix}$$

The resulting yield vector shows that height class 3 should be harvested to maximize the annual sustainable yield, since yl_3 = $2971.43 is the maximum entry.

Purge the user variables created in this problem section.

{'P1''P2''P3''P4''P5''GG1''GG2''GG3' 'GG4''GG5''CG''P''S''G1''G2''G3''G4' [PURGE]

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